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Geometric structures, Gromov norm and Kodaira dimensions

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ABSTRACT

We define the Kodaira dimension for 3-dimensional manifolds through Thurston's eight geometries, along with a classification in terms of this Kodaira dimension. We show this is compatible with other existing Kodaira dimensions and the partial order defined by non-zero degree maps. For higher dimensions, we explore the relations of geometric structures and mapping orders with various Kodaira dimensions and other invariants. Especially, we show that a closed geometric 4-manifold has nonvanishing Gromov norm if and only if it has geometry $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{H}^4$.

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1. Introduction

Complex Kodaira dimension $\kappa^h(M,J)$ provides a very successful classification scheme for complex manifolds. This notion is generalized by several authors (*cf.* [25,27,30,38,39]) to symplectic manifolds, especially of dimensions two and four. In these two dimensions, this symplectic Kodaira dimension is independent of the choice of symplectic structures [30]. In other words, it is a smooth invariant of the manifold which is thus denoted by $\kappa^s(M)$. In dimension four, the smaller the symplectic Kodaira dimension, the more we know. Symplectic 4-manifolds with $\kappa^s = -\infty$ are diffeomorphic to rational or ruled surfaces [35]. When $\kappa^s = 0$, all known examples are K3 surface, Enrique surface and T^2 bundles over T^2. Moreover, it is shown in [30] that a symplectic manifold with $\kappa^s = 0$ has the same homological invariants as one of the manifolds listed above. When $\kappa^s = 1$ or 2, no classification is possible since symplectic manifolds in both categories could admit arbitrary finitely presented group as their fundamental group [19].

In [9], the authors prove that complex and symplectic Kodaira dimensions are compatible with each other. More precisely, when a 4-manifold M admits at the same time both complex and symplectic structures (but the structures are not necessarily compatible with each other), then $\kappa^s(M) = \kappa^h(M,J)$. In [34], a general framework of "additivity of Kodaira dimension" is provided to further understand the compatibility of various Kodaira dimensions in possibly different dimensions. In particular, it is shown that the Kodaira dimensions are additive for fiber bundles, Lefschetz fibrations and coverings.

Higher dimensional generalizations of Kodaira dimension, *e.g.* symplectic Kodaira dimension in dimension six or higher, are less understood except for a proposed definition in [32]. Like complex Kodaira dimension, it will no longer be a smooth invariant. Hence, the study of this notion in higher dimensions will be associated to the study of deformation classes of symplectic structures and symplectic birational geometry.

As suggested by the additivity framework, dimension three should also attach certain counterpart of Kodaira dimension. In this paper, we give a definition of Kodaira dimension $\kappa^t(M)$ in dimension three through Thurston's eight 3-dimensional geometries and the Geometrization Theorem. It takes value from $-\infty$, 0, 1 or $\frac{3}{2}$. The half-integer $\frac{3}{2}$ is a new phenomenon, since complex or symplectic Kodaira dimension only takes value from integers. Certain classification with respect to $\kappa^t(M)$ is given. This notion is then discussed in the framework of "additivity of Kodaira dimension". In this sense,