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# A direct method of moving planes for the fractional Laplacian

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ABSTRACT

In this paper, we develop a direct *method of moving planes* for the fractional Laplacian. Instead of using the conventional extension method introduced by Caffarelli and Silvestre, we work directly on the non-local operator. Using the integral defining the fractional Laplacian, by an elementary approach, we first obtain the key ingredients needed in the *method of moving planes* either in a bounded domain or in the whole space, such as *strong maximum principles for anti-symmetric functions*, *narrow region principles*, and *decay at infinity*. Then, using simple examples, semi-linear equations involving the fractional Laplacian, we illustrate how this new *method of moving planes* can be employed to obtain symmetry and non-existence of positive solutions.
We firmly believe that the ideas and methods introduced here can be conveniently applied to study a variety of nonlocal

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problems with more general operators and more general nonlinearities.



## 1. Introduction

The fractional Laplacian in $R^n$ is a nonlocal pseudo-differential operator, assuming the form

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \lim_{\epsilon \to 0} \int\limits_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x)-u(z)}{|x-z|^{n+\alpha}} dz, \tag{1}$$

where $\alpha$ is any real number between 0 and 2. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^n$. In this space, it can also be equivalently defined in terms of the Fourier transform

$$\widehat{(-\Delta)^{\alpha/2}}u(\xi) = |\xi|^\alpha \hat{u}(\xi),$$

where $\hat{u}$ is the Fourier transform of $u$. One can extend this operator to a wider space of functions.

Let

$$L_\alpha = \{u : \mathbb{R}^n \to \mathbb{R} \mid \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+\alpha}}\, dx < \infty\}.$$

Then it is easy to verify that for $u \in L_\alpha \cap C^{1,1}_{loc}$, the integral on the right hand side of (1) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

The non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [2] introduced the *extension method* that reduced this nonlocal problem into a local one in higher dimensions. For a function $u : \mathbb{R}^n \to \mathbb{R}$, consider the extension $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases} div(y^{1-\alpha}\nabla U) = 0, & (x,y) \in \mathbb{R}^n \times [0,\infty), \\ U(x,0) = u(x). \end{cases}$$

Then

$$(-\triangle)^{\alpha/2}u(x) = -C_{n,\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \quad x \in \mathbb{R}^n.$$