

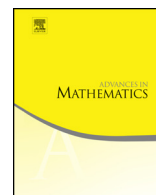


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# A direct method of moving planes for the fractional Laplacian

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## ABSTRACT

In this paper, we develop a direct *method of moving planes* for the fractional Laplacian. Instead of using the conventional extension method introduced by Caffarelli and Silvestre, we work directly on the non-local operator. Using the integral defining the fractional Laplacian, by an elementary approach, we first obtain the key ingredients needed in the *method of moving planes* either in a bounded domain or in the whole space, such as *strong maximum principles for anti-symmetric functions*, *narrow region principles*, and *decay at infinity*. Then, using simple examples, semi-linear equations involving the fractional Laplacian, we illustrate how this new *method of moving planes* can be employed to obtain symmetry and non-existence of positive solutions.

We firmly believe that the ideas and methods introduced here can be conveniently applied to study a variety of nonlocal

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problems with more general operators and more general nonlinearities.

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### 1. Introduction

The fractional Laplacian in  $\mathbb{R}^n$  is a nonlocal pseudo-differential operator, assuming the form

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz, \tag{1}$$

where  $\alpha$  is any real number between 0 and 2. This operator is well defined in  $\mathcal{S}$ , the Schwartz space of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^n$ . In this space, it can also be equivalently defined in terms of the Fourier transform

$$\widehat{(-\Delta)^{\alpha/2}u}(\xi) = |\xi|^\alpha \hat{u}(\xi),$$

where  $\hat{u}$  is the Fourier transform of  $u$ . One can extend this operator to a wider space of functions.

Let

$$L_\alpha = \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \right\}.$$

Then it is easy to verify that for  $u \in L_\alpha \cap C_{loc}^{1,1}$ , the integral on the right hand side of (1) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

The non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [2] introduced the *extension method* that reduced this nonlocal problem into a local one in higher dimensions. For a function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ , consider the extension  $U : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$  that satisfies

$$\begin{cases} \operatorname{div}(y^{1-\alpha} \nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x). \end{cases}$$

Then

$$(-\Delta)^{\alpha/2}u(x) = -C_{n,\alpha} \lim_{y \rightarrow 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \quad x \in \mathbb{R}^n.$$

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