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Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups ☆

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A B S T R A C T

We propose the analogues of boundary layer potentials for the sub-Laplacian on homogeneous Carnot groups/stratified Lie groups and prove continuity results for them. In particular, we show continuity of the single layer potential and establish the Plemelj type jump relations for the double layer potential. We prove sub-Laplacian adapted versions of the Stokes theorem as well as of Green's first and second formulae on homogeneous Carnot groups. Several applications to boundary value problems are given. As another consequence, we derive formulae for traces of the Newton potential for the sub-Laplacian to piecewise smooth surfaces. Using this we construct and study a nonlocal boundary value problem for the sub-Laplacian extending to the setting of the homogeneous Carnot groups M. Kac's "principle of not feeling the boundary". We also obtain similar results for higher powers of the sub-Laplacian. Finally, as another application, we prove refined versions of Hardy's inequality and of the uncertainty principle.

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Contents

1. Introduction

The central idea of solving boundary value problems for differential equations in a domain requires the knowledge of the corresponding fundamental solutions, and this idea has a long history dating back to the works of mathematicians such as Gauss [21,22] and Green [27]. Nowadays the appearing boundary layer operators and elements of the potential theory serve as major tools for the analysis and construction of solutions to boundary value problems. There is vast literature concerning modern theory of boundary layer operators and potential theory as well as their important applications. In addition, in last decades many interesting and promising works combining the group theory with the analysis of partial differential equations have been presented by many authors. For example, nilpotent Lie groups play an important role in deriving sharp subelliptic estimates for differential operators on manifolds, starting from the seminal paper by Rothschild and Stein [37]. Moreover, in recent decades, there is a rapidly growing interest for sub-Laplacians on Carnot groups (and also for operators on graded Lie groups), because these operators appear not only in theoretical settings (see e.g. Gromov [28] or Danielli, Garofalo and Nhieu [9] for general expositions from different points of view), but also in application settings such as mathematical models of crystal material and human vision (see, for example, [6] and [7]). Moreover, sub-Laplacians on homogeneous Carnot groups serve as approximations for general Hörmander's sums of squares of vector fields on manifolds in view of the Rothschild–Stein lifting theorem [37] (see also [16,38]).

In this paper we discuss elements of the potential theory and the theory of boundary layer operators on homogeneous Carnot groups. As we are not relying on the use of the control distance but on the fundamental solutions everything remains exactly the same (without any changes) if we replace the words 'homogeneous Carnot group' by 'stratified Lie group'. However, as a larger part of the current literature seems to use the former terminology we also adopt it for this paper.

From a different point of view than ours similar problems have been considered by Folland and Stein [17], Geller [24], Jerison [29], Romero [36], Capogna, Garofalo and