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The large scale geometry of strongly aperiodic subshifts of finite type

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ABSTRACT

A subshift on a group G is a closed, G-invariant subset of A^G, for some finite set A. It is said to be a subshift of finite type (SFT) if it is defined by a finite collection of "forbidden patterns", to be strongly aperiodic if all point stabilizers are trivial, and weakly aperiodic if all point stabilizers are infinite index in G. We show that groups with at least 2 ends have a strongly aperiodic SFT, and that having such an SFT is a QI invariant for finitely presented groups. We show that a finitely presented torsion free group with no weakly aperiodic SFT must be QI-rigid. The domino problem on G asks whether the SFT specified by a given set of forbidden patterns is empty. We show that decidability of the domino problem is a QI invariant.

1. Introduction

Recall that a topological dynamical system is a pair (Ω, G) where G is a group acting by homeomorphisms on the compact space Ω. For instance, if A is a finite discrete set, then the group G acts (on the right) on the compact space A^G by homeomorpisms via

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$$(\sigma \cdot h)(g) = \sigma(hg).$$

This action makes the pair (A^G, G) into a topological dynamical system called the right shift. When $G = \mathbb{Z}$, an element h of G acts on a biinfinite word $\sigma \in A^G$ by "shifting" it, whence the name. A closed, G-invariant subset of A^G is known as a subshift (for reference, see [7] or [8]). To say that a subshift X codes a dynamical system (Ω, G) means that there exists a continuous G-equivariant surjection from X to Ω.

Subshifts of finite type (see [8, §2]). How would one construct a subshift? The simplest idea is to start with a closed set C of A^G and intersect its G-translates. The most important case of this construction arises when C is determined by finitely many coordinates.

Definition 1.1. Let A be a finite set and G a group. If S is a finite subset of G and L a subset of A^S, then the clopen set

$$\{\sigma \in A^G : \sigma|_S \notin L\}$$

is known as a cylinder set. If C is a cylinder set, then the set X given by $\bigcap_{g\in G}(C \cdot g)$ is called a subshift of finite type. We say that X is defined on S. If F is a finite set, then $\alpha \in A^F$ is called a forbidden pattern for X if it is never equal to $(\sigma \cdot g)|_F$ for any $\sigma \in X$.

We say that a subshift of finite type $X \subseteq A^G$ is defined by a finite collection $\mathcal{F}$ of forbidden patterns $\alpha_i : F_i \to A$ if X is exactly the set of $\sigma \in A^G$ such that $(\sigma \cdot g)|_{F_i}$ is not equal to α_i for any i and any $g \in G$—i.e., if

$$X = \bigcap_{g\in G} \bigcap_{\alpha\in\mathcal{F}} \{\sigma \in A^G : \sigma|_{gF} \neq \alpha\} \cdot g.$$

1.1. Aperiodicity and the domino problem

Given a finite set of forbidden patterns $\mathcal{F}$, it is entirely possible that the subshift of finite type defined by $\mathcal{F}$ is empty.

Definition 1.2. Let G be a finitely generated group. We say that G has decidable domino problem if there exists an algorithm which takes as input a finite set of forbidden patterns $\mathcal{F}$ and determines whether the subshift they define is empty.

The domino problem for $\mathbb{Z}$. Suppose we are given a finite set of forbidden patterns $\mathcal{F}$ defining a subshift of finite type $X_\mathcal{F}$ over $\mathbb{Z}$. By compactness $X_\mathcal{F}$ is empty only if there is some n such that every A-coloring of the n-ball in $\mathbb{Z}$ includes some forbidden pattern. Hence, if $X_\mathcal{F}$ is empty, a Turing machine may discover this fact in finite time.

On the other hand, if $X_\mathcal{F}$ is nonempty, one would like to certify nonemptiness by finding a "constructible" $\sigma \in A^G$ which can be proven to lie in $X_\mathcal{F}$. The simplest constructible elements of $A^\mathbb{Z}$ are periodic—i.e., fixed by translation by some n and hence of