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Algebraic vertices of non-convex polyhedra

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A B S T R A C T

In this article we define an **algebraic vertex** of a generalized polyhedron and show that the set of algebraic vertices is the smallest set of points needed to define the polyhedron. We prove that the indicator function of a generalized polytope P is a linear combination of indicator functions of simplices whose vertices are algebraic vertices of P. We also show that the indicator function of any generalized polyhedron is a linear combination, with integer coefficients, of indicator functions of cones with apices at algebraic vertices and line-cones.
The concept of an algebraic vertex is closely related to the Fourier–Laplace transform. We show that a point $\mathbf{v}$ is an algebraic vertex of a generalized polyhedron P if and only if the tangent cone of P, at $\mathbf{v}$, has non-zero Fourier–Laplace transform.

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1. Introduction

We study the vertices of **non-convex polyhedra**, which we also call **generalized polyhedra**, and which we define as the finite union of convex polyhedra in $\mathbf{R}^d$.

There are many different ways to define a vertex of a generalized polyhedron P, most of them based on properties of the tangent cone to P at a point $\mathbf{v} \in P$. The tangent cone at $\mathbf{v}$, which we write as $\mathrm{tcone}(P, \mathbf{v})$, is intuitively the collection of all directions that we can 'see' if we stand at $\mathbf{v}$ and look into P (see Section 2 for a rigorous definition). We furthermore define a **line-cone** to be a cone that is the union of parallel lines.

One approach is to say that a point $\mathbf{v}$ is a vertex of a generalized polyhedron P if its tangent cone is not a **line-cone**. We call such a point $\mathbf{v}$ a **geometric vertex** of a generalized polyhedron, and it is clear that for a *convex* polyhedron this definition coincides with the usual definition of vertices (see the last Section). In this article we focus on another definition of vertices.

Definition 1. For a generalized polyhedron P, a point $\mathbf{v} \in P$ is called an **algebraic vertex** of P if the indicator function of its tangent cone $\mathrm{tcone}(P, \mathbf{v})$ cannot be represented (up to a set of measure zero) as a linear combination of indicator functions of line-cones.

The theorem of D. Frettlöh and A. Glazyrin [4] states that the indicator function of a convex cone which is not a line-cone cannot be represented as a sum of indicator functions of line-cones, implying that the vertices of an ordinary convex polytope are indeed algebraic vertices.

Our main result is the following description of algebraic vertices, showing that in some sense these generalized vertices form a minimal set of points needed to describe a **generalized polytope**, which is by definition a bounded generalized polyhedron.

Throughout, we denote the **indicator function** of any set $S \subset \mathbf{R}^d$ by $[S]$. In other words $[S](x) = 1$ if $x \in S$, and $[S](x) = 0$ if $x \notin S$.

Theorem 1. *Let $\mathcal{V}_P$ be the set of algebraic vertices of a generalized polytope $P \subset \mathbf{R}^d$, and let $\mathcal{T}_P$ be the set of simplices whose vertices lie in $\mathcal{V}_P$. Then*

$$[P] = \sum_{T_i \in \mathcal{T}_P} \alpha_i [T_i], \tag{1}$$

where the α_i are integers and the equality holds throughout $\mathbf{R}^d$, except perhaps for a set of measure zero.

Moreover, if $[P]$ is represented (up to measure zero) as a linear combination of indicator functions of some finite number of simplices, then the set of vertices of these simplices must contain $\mathcal{V}_P$.

It seems that weaker versions of this theorem were known for a long time, and in particular A. Gaifullin [5] showed that the indicator function of a polytope P is a lin-