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Isometry-invariant geodesics and the fundamental group, II

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ABSTRACT

We show that on a closed Riemannian manifold with fundamental group isomorphic to $\mathbb{Z}$, other than the circle, every isometry that is homotopic to the identity possesses infinitely many invariant geodesics. This completes a recent result in [20] of the second author.

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1. Introduction

In this paper, we complete the study began in [20] of isometry-invariant geodesics on closed Riemannian manifolds with infinite abelian fundamental group. Isometry-invariant geodesics play the role of closed geodesics in a Riemannian setting with symmetry. Given an isometry I of a closed connected Riemannian manifold (M, g), a geodesic $\gamma : \mathbb{R} \looparrowright M$ is called I-invariant if $I(\gamma(t)) = \gamma(t + \tau)$ for some positive $\tau > 0$ and for all $t \in \mathbb{R}$. Intuitively, these curves should be the closed geodesics of the possibly singular quotient M/I.

The study of isometry-invariant geodesics was initiated by Grove [11,12] in the 1970s. The problem admits a variational description, which generalizes the one of closed geodesics: isometry-invariant geodesics are the critical points of an energy function defined on a space of invariant paths. If the considered isometry is homotopic to the identity, this space of invariant paths is homotopy equivalent to the free loop space. This may induce someone to naively conjecture that all multiplicity results for closed geodesics remain true for isometry-invariant geodesics, provided the isometry is homotopic to the identity. A quite sophisticated argument due to Grove and Tanaka [14,15,13] shows that this is the case for Gromoll and Meyer's theorem: every closed Riemannian manifold with non-monogenic rational cohomology admits infinitely many isometry-invariant geodesics. This result is proved by cleverly exploiting the richness of the homology of the free loop space. However, there are multiplicity results, such as the existence of infinitely many closed geodesics on Riemannian 2-spheres [4,9,16], whose proofs need arguments that go beyond the abundance of the homology of the free loop space. These results may fail for isometry-invariant geodesics: for instance, a non-trivial rotation on a round 2-sphere has only one invariant geodesic.

A famous theorem of Bangert and Hingston implies that closed Riemannian manifolds with infinite abelian fundamental group always possess infinitely many closed geodesics. As in the case of the 2-sphere, the proof of this result combines general minimax tech-