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# On entropy for autoequivalences of the derived category of curves

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ABSTRACT

To an exact endofunctor of a triangulated category with a split-generator, the notion of entropy is given by Dimitrov–Haiden–Katzarkov–Kontsevich, which is a (possibly negative infinite) real-valued function of a real variable. It is important to evaluate the value of the entropy at zero in relation to the topological entropy. In this paper, we study the entropy at zero of an autoequivalence of the derived category of a complex smooth projective curve, and prove that it coincides with the natural logarithm of the spectral radius of the induced automorphism on its numerical Grothendieck group.


## 1. Introduction

To an exact endofunctor of a triangulated category with a split-generator, the notion of entropy is given by Dimitrov–Haiden–Katzarkov–Kontsevich in [2]. It is a (possibly negative infinite) real-valued function of a real variable motivated by an analogy with the topological entropy. It is known that the topological entropy of a surjective holomorphic endomorphism of a compact Kähler manifold coincides with the natural logarithm of the spectral radius of the induced action on the cohomology, which is the fundamental

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theorem of Gromov–Yomdin [4,3,12] (see also Theorem 3.6 in [9]). Concerning the entropy at zero of an exact endofunctor of a complex smooth proper variety, there is a lower bound, under a certain technical assumption, by the natural logarithm of the spectral radius of the induced action on the Hochschild homology (Theorem 2.9 in [2]). If the variety is projective, then the categorical and the topological entropies of surjective endomorphisms satisfying the assumption coincide (Theorem 2.12 in [2]). It is indeed possible to show this without the technical assumption used in [2], which is given in [7].

In this paper, we study the entropy of autoequivalences of the derived category of a smooth projective curve, and prove the following theorem, which is a natural generalization of the fundamental theorem of Gromov–Yomdin.

**Theorem 1.1** *(Theorem 3.1). Let $C$ be a complex smooth projective curve and $F$ an autoequivalence of the bounded derived category $\mathcal{D}^b(C)$ of coherent sheaves on $C$. The entropy $h(F)$ coincides with the natural logarithm of the spectral radius $\rho([F])$ of the induced automorphism $[F]$ on the numerical Grothendieck group $\mathcal{N}(C)$ of $\mathcal{D}^b(C)$. In particular, $\rho([F])$ is an algebraic number.*

The contents of this paper is as follows. In Section 2, we recall the definition and some basic properties of the entropy of exact endofunctors by [2]. We shall study the entropy for an autoequivalence of the bounded derived category of coherent sheaves on a smooth projective curve in Section 3. The non-trivial case is when a curve is an elliptic curve. We shall introduce the notion of the autoequivalence of type-$\mathbf{m}$ (Definition 3.7), which behaves very well in two points: it gives a representative of a conjugacy class of the automorphism group of the numerical Grothendieck group preserving the Euler form (Proposition 3.8), and it enables us to compute explicitly the entropy (Proposition 3.9). In Section 4, we shall recall the notion of LLS-period and give a proof of Proposition 3.8, the key proposition for our main theorem.

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## 2. Preliminaries

### *2.1. Notations and terminologies*

Throughout this paper, we work over the base field $\mathbb{C}$ and all triangulated categories are $\mathbb{C}$-linear and not equivalent to the zero category. The translation functor on a triangulated category is denoted by [1].

A triangulated category $\mathcal{T}$ is called *split-closed* if every idempotent in $\mathcal{T}$ splits, namely, if it contains all direct summands of its objects, and it is called *thick* if it is split-closed