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# Spectral and scattering theory for differential and Hankel operators

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ABSTRACT

We consider a class of Hankel operators $H$ realized in the space $L^2(\mathbb{R}_+)$ as integral operators with kernels $h(t+s)$ where $h(t) = P(\ln t)t^{-1}$ and $P(X) = X^n + p_{n-1}X^{n-1} + \cdots + p_0$ is an arbitrary real polynomial of degree $n$. This class contains the classical Carleman operator when $n = 0$. We show that a Hankel operator $H$ in this class can be reduced by an *explicit* unitary transformation (essentially by the Mellin transform) to a differential operator $A = vQ(D)v$ in the space $L^2(\mathbb{R})$. Here $Q(X) = X^n + q_{n-1}X^{n-1} + \cdots + q_0$ is a polynomial determined by $P(X)$ and $v(\xi) = \pi^{1/2}(\cosh(\pi\xi))^{-1/2}$ is the universal function. Then the operator $A = vQ(D)v$ reduces by the generalized Liouville transform to the standard differential operator $B = D^n + b_{n-1}(x)D^{n-1} + \cdots + b_0(x)$ with the coefficients $b_m(x)$, $m = 0, \ldots, n-1$, decaying sufficiently rapidly as $|x| \to \infty$. This allows us to use the results of spectral theory of differential operators for the study of spectral properties of generalized Carleman operators. In particular, we show that the absolutely continuous spectrum of $H$ is simple and coincides with $\mathbb{R}$ if $n$ is odd, and it has multiplicity 2 and coincides with $[0,\infty)$ if $n \geq 2$ is even. The singular continuous spectrum of $H$ is empty, and its eigenvalues may accumulate to the point 0 only. As a by-product of our considerations, we develop spectral theory of a new class of *degenerate* differential operators $A = vQ(D)v$

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where $Q(X)$ is an arbitrary real polynomial and $v(\xi)$ is an almost arbitrary real function decaying at infinity.


## 1. Introduction

**1.1.** Hankel operators can be defined by the formula

$$(Hu)(t) = \int_0^\infty h(t+s)u(s)ds \tag{1.1}$$

in the space $L^2(\mathbb{R}_+)$; thus integral kernels $h$ of Hankel operators depend on the sum of variables only. We refer to the books [16,17] for basic information on Hankel operators. Of course $H$ is symmetric if $h(t) = \overline{h(t)}$.

The spectra of bounded self-adjoint Hankel operators were characterized in the fundamental paper [12]. It was shown in [12] that the spectrum of a Hankel operator contains the point zero, and if zero is an eigenvalue, then necessary it has infinite multiplicity. Moreover, the spectral multiplicities of the points $\lambda$ and $-\lambda$ cannot differ by more than 2 and they cannot differ by more than 1 on the singular spectrum. Conversely, if the spectral measure and the multiplicity function of a self-adjoint operator $H$ possess these properties, then $H$ is unitarily equivalent to a Hankel operator. Since this result applies to all self-adjoint Hankel operators, it does not allow one to find spectral properties of specific classes of Hankel operators.

The cases where Hankel operators can be explicitly diagonalized are very scarce. The simplest and most important kernel $h(t) = t^{-1}$ was considered by T. Carleman in [4]. The eigenfunctions of the continuous spectrum $\theta(t,k)$, $k \in \mathbb{R}$, of this operator are given by the formula $\theta(t,k) = t^{-1/2+ik}$. They satisfy the equation $H\theta(k) = \lambda(k)\theta(k)$ with the dispersion relation $\lambda(k) = \pi\big(\cosh(\pi k)\big)^{-1}$. Thus the spectrum of $H$ is absolutely continuous, it has multiplicity 2 and coincides with the interval $[0,\pi]$. It was pointed out by J.S. Howland in [9] that there is a somewhat mysterious affinity between Hankel and differential operators. In terms of this analogy, the Carleman operator plays the role of the operator $D^2$ in the space $L^2(\mathbb{R})$.

The results on the Carleman operator can be extended to more complicated kernels. We note the classical papers [13] by F. Mehler who considered the kernel $h(t) = (t+1)^{-1}$ and [11,19] by W. Magnus and M. Rosenblum who considered the kernel $h(t) = t^{-1}e^{-t}$ (M. Rosenblum considered also more general kernels with the same singularity at $t = 0$). The corresponding Hankel operators $H$ were diagonalized in terms of the Legendre and Whittaker functions, respectively. The spectrum of these operators is absolutely continuous, simple, and it coincides with the interval $[0,\pi]$. These results can be deduced from