Advances in Mathematics 308 (2017) 767–814

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

A spectral sequence for polyhedral products

A. Bahri [a], M. Bendersky [b,*], F.R. Cohen [c], S. Gitler

[a] *Department of Mathematics, Rider University, Lawrenceville, NJ 08648, USA*
[b] *Department of Mathematics CUNY, East 695 Park Avenue, New York, NY 10065, USA*
[c] *Department of Mathematics, University of Rochester, Rochester, NY 14625, USA*

ARTICLE INFO

Article history:
Received 23 April 2016
Received in revised form 6 October 2016
Accepted 24 December 2016
Communicated by Henning Krause

This paper is dedicated to our friend and colleague Samuel Gitler

MSC:
primary 55T05, 55T25, 55N10, 55N35
secondary 52C35

Keywords:
Polyhedral products
Cohomology
Spectral sequences
Stable splittings
Moment-angle complexes
Simplicial complexes

ABSTRACT

The purpose of this paper is to exhibit fine structure for polyhedral products $Z(K;(\underline{X},\underline{A}))$ and polyhedral smash products $\widehat{Z}(K;(\underline{X},\underline{A}))$ where moment-angle complexes are special cases in which $(X,A) = (D^2,S^1)$. There are three main parts to this paper.

(1) One part gives a natural filtration of the polyhedral product together with properties of the resulting spectral sequence in Theorem 2.15. Applications of this spectral sequence are given.
(2) The second part gives a homological decomposition of $\widehat{Z}(K;(\underline{X},\underline{A}))$ in case $(\underline{X},\underline{A})$ consists of CW pairs by using part 1.
(3) Applications to the ring structure of $Z(K;(\underline{X},\underline{A}))$ are given for CW-pairs (X,A) satisfying suitable freeness conditions.

* Corresponding author.
E-mail addresses: bahri@rider.edu (A. Bahri), mbenders@hunter.cuny.edu (M. Bendersky), fred.cohen@rochester.edu (F.R. Cohen).

http://dx.doi.org/10.1016/j.aim.2016.12.028
0001-8708/

1. Introduction

The subject of this paper is the homology of polyhedral products $Z(K;(\underline{X},\underline{A}))$, and polyhedral smash products $\widehat{Z}(K;(\underline{X},\underline{A}))$ [9,5,1–3]. Definitions are listed in section 2 of this paper.

One of the purposes of this article is to give the Hilbert–Poincaré series for the polyhedral product $Z(K;(\underline{X},\underline{A}))$ in terms of

(1) the kernel, image, and cokernel of the induced maps

$$H^*(X_i) \to H^*(A_i)$$

for all i, and

(2) the full sub-complexes of K.

This computation was also worked out in [4][1] using more geometric methods.

This is achieved by analysis of a spectral sequence abutting to the cohomology of the polyhedral product $Z(K;(\underline{X},\underline{A}))$ by filtering this space with the left-lexicographical ordering of simplices. The method applies to a generalized multiplicative cohomology theory, h^* as well. The spectral sequence is then used to describe some features of the ring structure of $h^*(Z(K;(\underline{X},\underline{A})))$.

Qibing Zheng [12] gives an alternative description of the cohomology of a polyhedral product. Our methods are distinct from his and the presentation of the computational results assumes a different form. Unlike the spectral sequence developed here, his collapses at the E_2 term.

2. Definitions, and main results

The basic constructions addressed in this article are defined in this section. First recall the definition of an abstract simplicial complex.

Definition 2.1.

(1) Let K denote an abstract simplicial complex with m vertices labeled by the set $[m]=\{1,2,\ldots,m\}$. Thus, K is a subset of the power set of $[m]$ such that an element given by a $(k-1)$-simplex σ of K is given by an ordered sequence $\sigma=(i_1,\ldots,i_k)$ with $1\le i_1<\cdots<i_k\le m$ such that if $\tau\subset\sigma$, then τ is a simplex of K. In particular the empty set $\emptyset$ is a subset of σ and so it is in K. The vertex set of σ $\{i_1,\ldots,i_k\}$ will be denoted $v(\sigma)$.

[1] While the third and fourth author were visiting the Institute for Advanced Study.