Advances in Mathematics 308 (2017) 815–835

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Asymptotic behavior of the dimension of the Chow variety

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A R T I C L E I N F O

Article history:
Received 20 January 2015
Received in revised form 21 December 2016
Accepted 21 December 2016
Communicated by Ravi Vakil

Keywords:
Chow variety
Cycle
Volume

A B S T R A C T

First, we calculate the dimension of the Chow variety of degree d cycles on projective space. We show that the component of maximal dimension usually parametrizes degenerate cycles, confirming a conjecture of Eisenbud and Harris. Second, for a numerical class α on an arbitrary variety, we study how the dimension of the components of the Chow variety parametrizing cycles of class $m\alpha$ grows as we increase m. We show that when the maximal growth rate is achieved, α is represented by cycles that are "degenerate" in a precise sense.

1. Introduction

Let X be an integral projective variety over an algebraically closed field. Fix a numerical cycle class α on X and let $\mathrm{Chow}(X, \alpha)$ denote the components of $\mathrm{Chow}(X)$ which parametrize cycles of class α. Our goal is to study the dimension of $\mathrm{Chow}(X, \alpha)$ and its relationship with the geometry and positivity of the cycles representing α. In general one expects that the maximal components of Chow should parametrize subvarieties that are "degenerate" in some sense, and our results verify this principle in a general setting.

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[1] The author is supported by NSF Award 1004363.

http://dx.doi.org/10.1016/j.aim.2016.12.024

We first consider the most important example: projective space. [2] analyzes the dimension of the Chow variety of curves on $\mathbb{P}^n$. Let ℓ denote the class of a line in $\mathbb{P}^n$. Then for $d > 1$,

$$\dim \mathrm{Chow}(\mathbb{P}^n, d\ell) = \max\left\{2d(n-1), \frac{d^2+3d}{2} + 3(n-2)\right\}.$$

The first number is the dimension of the space of unions of d lines on $\mathbb{P}^n$, and the second is the dimension of the space of degree d planar curves. Note the basic dichotomy: in low degrees the maximal dimension is achieved by unions of linear spaces, while for sufficiently high degrees the maximal dimension is achieved by "maximally degenerate" irreducible curves.

We prove an analogous statement in arbitrary dimension, establishing a conjecture of [2]:

Theorem 1.1. *Let α denote the class of the k-plane on $\mathbb{P}^n$. Then for $d > 1$ the dimension of* $\mathrm{Chow}(\mathbb{P}^n, d\alpha)$ *is*

$$\max\left\{d(k+1)(n-k), \binom{d+k+1}{k+1} - 1 + (k+2)(n-k-1)\right\}.$$

The first number is the dimension of the space of unions of d k-planes on $\mathbb{P}^n$ and the second is the dimension of the space of degree d hypersurfaces in $(k+1)$-planes. The same dichotomy seen for curves also holds in higher dimensions. The approach is to reduce to the result of [2] by cutting down by hyperplane sections. It would be interesting to develop analogous results for other varieties with simple structure, e.g. quadrics or rational normal scrolls.

For arbitrary varieties X, it is too much to hope for a precise relationship between $\dim \mathrm{Chow}(X, \alpha)$ and the "degeneracy" of cycles. Instead, by analogy with the divisor case, we obtain a cleaner picture by studying the asymptotics of $\dim \mathrm{Chow}(X, m\alpha)$ as m increases. For a Cartier divisor L on a smooth variety X of dimension n, the asymptotic behavior of sections is controlled by an important invariant known as the volume:

$$\mathrm{vol}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X, \mathcal{O}_X(mL))}{m^n/n!}.$$

We formulate and study an analogous construction for arbitrary cycles.

The expected growth rate of $\dim \mathrm{Chow}(X, m\alpha)$ as m increases can be predicted by supposing that $m\alpha$ is a the pushforward of a divisor class on a fixed $(k+1)$-dimensional subvariety of X: Theorem 5.1 shows that $\dim \mathrm{Chow}(X, m\alpha) < Cm^{k+1}$ for some constant C. The variation function identifies the best possible constant C.

Definition 1.2. Let X be a projective variety and suppose $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \leq k < \dim X$. The variation of α is