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On isolated singularities with a noninvertible finite endomorphism

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ABSTRACT

We prove that if $\phi : (X, 0) \to (X, 0)$ is a finite endomorphism of an isolated singularity such that $\deg(\phi) \geq 2$ and ϕ is étale in codimension 1, then X is $\mathbb{Q}$-Gorenstein and log canonical.

1. Introduction

Let us start with an easy example. Let C be a smooth curve of genus g. By an argument using Riemann–Hurwitz Theorem, one can see that C has a finite endomorphism ϕ of degree ≥ 2 if and only if $g \leq 1$. In this case, assume that there is an ample divisor H on C such that $\phi^* H$ is a multiple of H. Then ϕ induces a finite endomorphism on the cone X over C with polarization mH, where m is a sufficiently large integer. On the other hand, one can see easily by adjunction that a normal cone over a smooth curve of genus

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g has log canonical singularity if and only if $g \leq 1$. This phenomenon is true in general. In this paper, we prove the following theorem:

Theorem 1.1. *Let $(X, 0)$ be a normal projective variety with isolated singularity $0 \in X$. Suppose that there exists a finite endomorphism $\phi : (X, 0) \to (X, 0)$ such that $\deg \phi \geq 2$ and ϕ is étale in codimension 1. Then X is $\mathbb{Q}$-Gorenstein and log canonical.*

The assumption that X has isolated singularity is necessary. Otherwise, let $X = E \times V$, where E is an elliptic curve and V is an arbitrary variety with a bad singularity. Then X has an induced noninvertible étale endomorphism from E.

We briefly review the history of this problem. For the definitions of related terminologies, we refer to Section 2 and [1]. The surface case is studied in [24]. Let X be a normal surface and $f : Y \to X$ be the minimal resolution. The relative Zariski decomposition yields $K_{Y/X} = P + N$. Wahl's invariant is defined as the nonnegative intersection number $-P^2$, which is the key ingredient in the study of surfaces with noninvertible finite endomorphisms. A classification of such surfaces is given in [7,9]. Wahl's invariant is generalized to higher dimensions by Boucksom, de Fernex and Favre [1]. Due to the absence of minimal resolutions, they consider log discrepancy divisors on all birational models over X as Shokurov's b-divisor $A_{\mathcal{X}/X}$. The Zariski decomposition is replaced by the nef envelope $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. It can be shown that $-(\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X}))^n$ is a well-defined finite nonnegative number, which is called $\mathrm{vol}_{\mathrm{BdFF}}(X)$. This volume behaves well under finite morphisms. In particular, they prove the following theorem:

Theorem 1.2. *[1, Theorem A and B], [13, Proposition 2.12] For normal isolated singularities $(X, 0)$ with noninvertible finite endomorphism, $\mathrm{vol}_{\mathrm{BdFF}}(X) = 0$. Moreover, when X is $\mathbb{Q}$-Gorenstein, $\mathrm{vol}_{\mathrm{BdFF}}(X) = 0$ if and only if X has log canonical singularity.*

The same theorem is obtained in [4] by analyzing the behavior of non-log-canonical centers under finite pullback. In [13], Fulger defines a courser volume $\mathrm{vol}_{\mathrm{F}}(X)$ as the asymptotic order of growth of plurigenera, which coincides with $\mathrm{vol}_{\mathrm{BdFF}}(X)$ when X is $\mathbb{Q}$-Gorenstein.

Unfortunately, in [26], the author produces a non-$\mathbb{Q}$-Gorenstein isolated singularity $(X, 0)$ such that $\mathrm{vol}_{\mathrm{BdFF}}(X) = 0$ while there is no boundary Δ such that (X, Δ) is log canonical. We should remark that, in this example, X admits a small log canonical modification [23], [4, Proposition 2.4].

The $\mathbb{Q}$-Gorenstein case is further studied in [26, Section 3]. Specifically, the author shows that, like the surface case, $\mathrm{vol}_{\mathrm{BdFF}}(X)$ can be calculated by an intersection number on a certain birational model $f : Y \to X$, namely, the log canonical modification [23]. A key property of such a model is that $K_Y + E_f$ is f-ample, where E_f is the reduced exceptional divisor. In the non-$\mathbb{Q}$-Gorenstein case, the existence of the log canonical modification is conjectured to be true assuming the full minimal model program including the abundance conjecture, but has not yet been proved.