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Bernstein inequality and holonomic modules ☆

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ABSTRACT

In this paper we study the representation theory of filtered algebras with commutative associated graded whose spectrum has finitely many symplectic leaves. Examples are provided by the algebras of global sections of quantizations of symplectic resolutions, quantum Hamiltonian reductions, and spherical symplectic reflection algebras. We introduce the notion of holonomic modules for such algebras. We show that, provided the algebraic fundamental groups of all leaves are finite, the generalized Bernstein inequality holds for the simple modules and turns into equality for holonomic simples. Under the same finiteness assumption, we prove that the associated variety of a simple holonomic module is equi-dimensional. We also prove that, if the regular bimodule has finite length, then any holonomic module has finite length. This allows one to reduce the Bernstein inequality for arbitrary modules to simple ones. We prove that the regular bimodule has finite length for the global sections of quantizations of symplectic resolutions, for quantum Hamiltonian reductions and for Rational Cherednik algebras. The paper contains a joint appendix by the author and Etingof that motivates the definition of a holonomic module in the case of global sections of a quantization of a symplectic resolution.

☆ The paper contains an appendix, which is joint with Pavel Etingof from MIT.
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1. Introduction

1.1. Holonomic modules

Let $\mathcal{A}$ be an infinite dimensional algebra. The study of all representations of $\mathcal{A}$ is a wild task even if $\mathcal{A}$ itself is relatively easy, e.g., $\mathcal{A}$ is the algebra $D(\mathbb{C}^n)$ of algebraic linear differential operators on $\mathbb{C}^n$ (a.k.a. the Weyl algebra of the symplectic vector space $\mathbb{C}^{2n}$). In this case (and in a more general case of the algebra $D(X)$ of differential operators on a smooth affine variety X or a sheaf D_X of differential operators on a non-necessarily affine smooth variety) there is a nice class of modules called *holonomic*. Namely, to a finitely generated D_X-module one can assign the so called singular support that is a closed subvariety of T^*X. This subvariety is always coisotropic. A D_X-module is called *holonomic* if its singular support is lagrangian. Many D_X-modules appearing in “nature” are holonomic.

One question addressed in the present paper is how to generalize the notion of a holonomic module to a wider class of algebras. In the D-module situation it is easy to see that any holonomic module has finite length. We will see that this is the case in many other situations.

1.2. Bernstein inequality

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of characteristic 0. We can form the universal enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W be the Weyl group. Then the center Z of $U(\mathfrak{g})$ is identified with $S(\mathfrak{h})^W$ via the so called Harish-Chandra isomorphism. So, for $\lambda \in \mathfrak{h}^*$, one can consider the central reduction U_λ of $U(\mathfrak{g})$, i.e., the quotient $U_\lambda := U(\mathfrak{g})/U(\mathfrak{g})S(\mathfrak{h})^W_\lambda$, where $S(\mathfrak{h})^W_\lambda$ stands for the maximal ideal in $S(\mathfrak{h})^W$ of all elements that vanish at λ. Then there is a remarkable property of U_λ known as the (generalized) Bernstein inequality:

$$\operatorname{GK-dim}(M) \geqslant \frac{1}{2} \operatorname{GK-dim}\left(U_\lambda / \operatorname{Ann}_{U_\lambda}(M)\right),$$

where GK- dim stands for the Gelfand–Kirillov (shortly, GK) dimension and $\operatorname{Ann}_{U_\lambda}(M)$ denotes the annihilator of M in U_λ, a two-sided ideal.

Another goal of this paper is to generalize the Bernstein inequality for a wider class of algebras. We will also see that the inequality becomes an equality for holonomic modules.

1.3. Algebras of interest

Our base field is the field $\mathbb{C}$ of complex numbers. Let $\mathcal{A} = \bigcup_{i\geqslant 0} \mathcal{A}_{\leqslant i}$ be a $\mathbb{Z}_{\geqslant 0}$-filtered associative $\mathbb{C}$-algebra such that the associated graded algebra $A := \operatorname{gr}\mathcal{A}$ is commutative and finitely generated. Let d be a positive integer such that $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-d}$. We get a natural degree $-d$ Poisson bracket on A. We assume that $X := \operatorname{Spec}(A)$ has finitely