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Infinite random matrices & ergodic decomposition of finite and infinite Hua–Pickrell measures

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ABSTRACT

The ergodic decomposition of a family of Hua–Pickrell measures on the space of infinite Hermitian matrices is studied. By combining previous results of Borodin–Olshanski and our new results, we obtain the first complete description of the ergodic decomposition of Hua–Pickrell measures. First, we show that the ergodic components of any Hua–Pickrell probability measure have no Gaussian factors. Secondly, we show that the sequence of asymptotic eigenvalues of Hua–Pickrell random matrices is balanced in a certain sense and has a "principal value" which coincides with the parameter that reflects the presence of Dirac factor in an ergodic component. This allows us to *identify* the ergodic decomposition of any Hua–Pickrell probability with a certain determinantal point process with hypergeometric kernel as introduced by Borodin–Olshanski. Finally, we extend the aforesaid results to the case of infinite Hua–Pickrell measures. By using the theory of infinite determinantal measures recently introduced by A.I. Bufetov, we are able to *identify* the ergodic decomposition of Hua–Pickrell infinite measure with a certain infinite determinantal measure. This resolves a problem of Borodin and Olshanski.

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1. Introduction

1.1. Main objects

Let H be the space of infinite Hermitian matrices, i.e.,

$$H = \Big\{ X = [X_{ij}]_{i,j=1}^{\infty} : X_{ij} \in \mathbb{C}, \overline{X}_{ij} = X_{ji} \Big\}.$$

Denote $U(\infty)$ the inductive limit of the unitary groups $U(N)$ with respect to the natural inclusions $U(N) \hookrightarrow U(N+1)$, equivalently, we may regard $U(\infty)$ as the group of infinite unitary matrices $U = [U_{ij}]_{i,j=1}^{\infty}$ with finitely many entries $U_{ij} \neq \delta_{ij}$. The group $U(\infty)$ acts on H by conjugation:

$$T_u X = uXu^{-1}.$$

The $U(\infty)$-invariant measures on the space H have been studied in many branches of mathematics such as the theory of random matrices, the representation theory of infinite dimensional groups, the theory of harmonic analysis on infinite dimensional groups, etc.

It is well known (see, e.g. [15]) that there exists a natural correspondence between the set of unitarily invariant probability measures on H and the set of spherical representations[2] of the infinite Cartan motion group

$$H(\infty) \rtimes U(\infty),$$

where $H(\infty)$ is the inductive limit of $H(N)$, the space of Hermitian matrices of size $N \times N$, with respect to the natural inclusions. In this correspondence, a $U(\infty)$-ergodic probability measure on H corresponds to an irreducible spherical representation of $H(\infty) \rtimes U(\infty)$. Since the groups $U(\infty)$ and $H(\infty) \rtimes U(\infty)$ are not locally compact, they do not admit Haar measures, this fact is a major obstacle to study the representations of such groups. The spherical representations of $H(\infty) \rtimes U(\infty)$ corresponding to some naturally defined invariant probability measures on H are analogues of regular representations. The decomposition of these spherical representations of $H(\infty) \rtimes U(\infty)$ into irreducible components can be achieved by obtaining an ergodic decomposition of corresponding invariant measures on H.

The main subject of this paper is a family of unitarily invariant measures $\{m^{(s)} : s \in \mathbb{C}\}$ defined on H, which are called Hua–Pickrell measures. When $\Re s > -\frac{1}{2}$, they are probability measures and when $\Re s \leq -\frac{1}{2}$, they have infinite mass.

Our goal will be twofold. First, following Borodin and Olshanski, we continue the study of the ergodic decomposition of these Hua–Pickrell probability measures (that is $\Re s > -\frac{1}{2}$). Our main result is, by combining previous results of Borodin–Olshanski and

[2] A unitary representation of $H(\infty) \rtimes U(\infty)$ on $\mathscr{H}$ is said to be spherical if there exists a $U(\infty)$-invariant cyclic vector in $\mathscr{H}$.