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On q-integrals over order polytopes ☆

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A B S T R A C T

A combinatorial study of multiple q-integrals is developed. This includes a q-volume of a convex polytope, which depends upon the order of q-integration. A multiple q-integral over an order polytope of a poset is interpreted as a generating function of linear extensions of the poset. Specific modifications of posets are shown to give predictable changes in q-integrals over their respective order polytopes. This method is used to combinatorially evaluate some generalized q-beta integrals. One such application is a combinatorial interpretation of a q-Selberg integral. New generating functions for generalized Gelfand–Tsetlin patterns and reverse plane partitions are established. A q-analogue to a well known result in Ehrhart theory is generalized using q-volumes and q-Ehrhart polynomials.

Contents

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1. Introduction

The main object in this paper is the q-integral

$$\int_0^1 f(x)d_qx = (1-q)\sum_{i=0}^{\infty} f(q^i)q^i,$$

which was introduced by Thomae [23] and Jackson [12]. The q-integral is a q-analogue of the Riemann integral. Fermat used it to evaluate $\int_0^1 x^n dx$. See [2, §10.1] for more details of the history of q-integrals. Many important integrals have q-analogues in terms of q-integrals, such as q-beta integrals and q-Selberg integrals. In this paper we develop combinatorial methods to study q-integrals.

The original motivation of this paper was to generalize Stanley's combinatorial interpretation of the Selberg integral [19]

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\le i<j\le n} |x_i-x_j|^{2\gamma} dx_1\cdots dx_n$$
$$= \prod_{j=1}^{n} \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)},$$

where n is a positive integer and α, β, γ are complex numbers such that $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\mathrm{Re}(\gamma) > -\min\{1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1)\}$. Stanley [22, Exercise 1.10 (b)] found a combinatorial interpretation of the above integral when $\alpha-1, \beta-1$ and 2γ are nonnegative integers in terms of permutations. His idea is to interpret the integral as the probability that a random permutation satisfies certain properties. This