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Subdyadic square functions and applications to weighted harmonic analysis ☆

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ABSTRACT

Through the study of novel variants of the classical Littlewood–Paley–Stein g-functions, we obtain pointwise estimates for broad classes of highly-singular Fourier multipliers on $\mathbb{R}^d$ satisfying regularity hypotheses adapted to fine (subdyadic) scales. In particular, this allows us to efficiently bound such multipliers by geometrically-defined maximal operators via general weighted L^2 inequalities, in the spirit of a well-known conjecture of Stein. Our framework applies to solution operators for dispersive PDE, such as the time-dependent free Schrödinger equation, and other highly oscillatory convolution operators that fall well beyond the scope of the Calderón–Zygmund theory.

1. Introduction

A common feature of many themes in both classical and contemporary harmonic analysis is the pivotal role played by operators which exhibit a certain *quadratic* structure. Such operators are usually referred to as *square functions*, and their study has its roots

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in the classical Littlewood–Paley theory (see for example [36,37,39]). A common role of the square functions is to capture manifestations of orthogonality in L^p spaces for $p \neq 2$. A primordial example is the square function (or *g-function*)

$$g(f)(x) := \left(\int_0^\infty \left| \frac{\partial u}{\partial t}(x,t) \right|^2 t dt \right)^{1/2}, \tag{1}$$

where $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ denotes the Poisson integral of the function f on $\mathbb{R}^d$. While Plancherel's theorem quickly reveals that $\|g(f)\|_2 \equiv \|f\|_2$, the key point is that this property essentially persists on L^p – that is, the norms $\|g(f)\|_p$ and $\|f\|_p$ are equivalent for all $1 < p < \infty$. Such facts have many important consequences, making square functions a central tool in modern analysis and PDE. In particular, square functions play a striking role in the classical theory of Fourier multipliers. On an abstract level, this approach to multipliers, originating in fundamental work of Stein [36], consists of identifying square functions g_1 and g_2 for which we have the *pointwise* estimate

$$g_1(T_m f)(x) \lesssim g_2(f)(x); \tag{2}$$

here T_m denotes the convolution operator with Fourier multiplier m.[1] Given such an estimate one may then deduce bounds on T_m from bounds on the square functions g_1 and g_2. More specifically, if one has

$$\|f\|_X \lesssim \|g_1(f)\|_Y \quad \text{and} \quad \|g_2(f)\|_Y \lesssim \|f\|_Z, \tag{3}$$

for suitable normed spaces X, Y, Z, then the pointwise estimate (2) quickly reveals that

$$\|T_m f\|_X \lesssim \|g_1(T_m f)\|_Y \lesssim \|g_2(f)\|_Y \lesssim \|f\|_Z; \tag{4}$$

that is, T_m is bounded from Z to X.[2] The prime example of this approach in action is Stein's celebrated proof of the classical Hörmander–Mikhlin multiplier theorem, which states that if a Fourier multiplier m on $\mathbb{R}^d$ satisfies

$$\sup_{r>0} \|m(r\cdot)\Psi\|_{H^\sigma} < \infty \tag{5}$$

for some $\sigma > d/2$, or equivalently

$$\sup_{r>0} r^\theta r^{-d/2} \|m\Psi(r^{-1}\cdot)\|_{\dot{H}^\theta} < \infty \tag{6}$$

[1] Throughout this paper we shall write $A \lesssim B$ if there exists a constant c such that $A \leq cB$. In particular, this constant will always be independent of the input function f, variable x and weight function w. The relations $A \gtrsim B$ and $A \sim B$ are defined similarly.

[2] Of course this requires that the norm $\|\cdot\|_Y$ is increasing in the sense that $f_1 \lesssim f_2 \Longrightarrow \|f_1\|_Y \lesssim \|f_2\|_Y$.