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Bernstein–Sato polynomials for maximal minors and sub-maximal Pfaffians

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ABSTRACT

We determine the Bernstein–Sato polynomials for the ideal of maximal minors of a generic $m \times n$ matrix, as well as for that of sub-maximal Pfaffians of a generic skew-symmetric matrix of odd size. As a corollary, we obtain that the Strong Monodromy Conjecture holds in these two cases.

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1. Introduction

Consider a polynomial ring $S = \mathbb{C}[x_1, \cdots, x_N]$ and let $\mathcal{D} = S[\partial_1, \cdots, \partial_N]$ denote the associated Weyl algebra of differential operators with polynomial coefficients ($\partial_i = \frac{\partial}{\partial x_i}$). For a non-zero element $f \in S$, the set of polynomials $b(s) \in \mathbb{C}[s]$ for which there exists a differential operator $P_b \in \mathcal{D}[s]$ such that

$$P_b \cdot f^{s+1} = b(s) \cdot f^s \tag{1.1}$$

form a non-zero ideal. The monic generator of this ideal is called the Bernstein–Sato polynomial (or the b-function) of f, and is denoted $b_f(s)$. The b-function gives a measure of the singularities of the scheme defined by $f = 0$, and its zeros are closely related to the eigenvalues of the monodromy on the cohomology of the Milnor fiber. In the case of a single hypersurface, its study has originated in [2,23], and later it has been extended to more general schemes in [5] (see Section 2.5). Despite much research, the calculation of b-functions remains notoriously difficult: several algorithms have been implemented to compute b-functions, and a number of examples have been worked out in the literature, but basic instances such as the b-functions for determinantal varieties are still not understood. In [3] and [4], Budur posed as a challenge and reviewed the progress on the problem of computing the b-function of the ideal of $p \times p$ minors of the generic $m \times n$ matrix. We solve the challenge for the case of maximal minors in this paper, and we also find the b-function for the ideal of $2n \times 2n$ Pfaffians of the generic skew-symmetric matrix of size $(2n+1) \times (2n+1)$. For maximal minors, our main result is as follows:

Theorem on Maximal Minors *(Theorem 4.1). Let $m \geq n$ be positive integers, consider the generic $m \times n$ matrix of indeterminates (x_{ij}), and let $I = I_n$ denote the ideal in the polynomial ring $S = \mathbb{C}[x_{ij}]$ which is generated by the $n \times n$ minors of (x_{ij}). The b-function of I is given by*

$$b_I(s) = \prod_{i=m-n+1}^{m} (s+i).$$

When $m = n$, I is generated by a single equation – the determinant of the generic $n \times n$ matrix – and the formula for $b_I(s)$ is well-known (see [17, Appendix] or [21, Section 5]). For general $m \geq n$, if we let $Z_{m,n}$ denote the zero locus of I, i.e. the variety of $m \times n$ matrices of rank at most $n-1$, then using the renormalization (2.28) our theorem states that the b-function of $Z_{m,n}$ is $\prod_{i=0}^{n-1}(s+i)$. It is interesting to note that this only depends on the value of n and not on m.

The statement of the Strong Monodromy Conjecture of Denef and Loeser [9] extends naturally from the case of one hypersurface to arbitrary ideals, and it asserts that the poles of the topological zeta function of I are roots of $b_I(s)$. We verify this conjecture for maximal minors and sub-maximal Pfaffians in Section 5. When $I = I_n$ is the ideal of