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Stable basic sets for finite special linear and unitary groups

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A B S T R A C T

In this paper we show, using Deligne–Lusztig theory and Kawanaka's theory of generalised Gelfand–Graev representations, that the decomposition matrix of the special linear and unitary group in non-defining characteristic can be made unitriangular with respect to a basic set that is stable under the action of automorphisms.

1. Introduction

While ordinary characters of finite groups of Lie type are fairly well known, not so much is known about Brauer characters and in particular we do not have a general parameterisation for them. Decomposition matrices offer information about Brauer characters, and the unitriangularity of such matrices sets up a natural bijection from the set of Brauer characters to a corresponding so-called basic set of ordinary characters (see Definition 2.2). When we have such a bijection, we can try to obtain an equivariant one with respect to automorphisms. This is the case if and only if the basic set is stable

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under the action of automorphisms (see Lemma 2.3). This is useful to deal with some counting conjectures (see [4, Theorem 7.4]).

The unitriangularity of the decomposition matrices of finite general linear groups $\mathrm{GL}_n(q)$ in non-defining characteristic ℓ was proved by Dipper in [6] and [7], while the case of special linear groups $\mathrm{SL}_n(q)$ was done by Kleshchev and Tiep in [14]. However the techniques used in these papers rely on an actual construction of Brauer characters of $\mathrm{GL}_n(q)$. Such a construction is not known in the case of general unitary groups $\mathrm{GU}_n(q)$ so these methods cannot be applied for now to prove the unitriangularity result for either $\mathrm{GU}_n(q)$ or the special unitary groups $\mathrm{SU}_n(q)$. Nevertheless the result has been shown for $\mathrm{GU}_n(q)$ by Geck in [9], using Kawanaka's theory of Generalised Gelfand Graev Representations (GGGRs for short). Such a method can also be used to recover the result for $\mathrm{GL}_n(q)$. In [10] the same method was applied in the case of $\mathrm{SU}_n(q)$, but only the cases where $\ell \nmid \gcd(n, q+1)$ could be treated (see [10, Theorem C]). In this case the usual basic set for $\mathrm{GU}_n(q)$ gives by restriction a basic set for $\mathrm{SU}_n(q)$. Investigating the methods developed in [14] and translating them in the context of Deligne–Lusztig theory and GGGRs, we first found that their methods could be adapted to $\mathrm{SU}_n(q)$. But as was the case for $\mathrm{SL}_n(q)$, the many basic sets obtained for $\mathrm{SU}_n(q)$ were not stable with respect to automorphisms. In this paper we prove the following stronger statement:

Theorem A. *Let q be a power of some prime number p and ℓ be a prime number not dividing q. Let $n \geq 2$ and let $G \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q)\}$ be either the special linear or unitary group over a finite field with q elements. Then G has a unitriangular basic set in characteristic ℓ that is stable under the action of* $\mathrm{Aut}(G)$.

Let $\tilde{G} = \mathrm{GL}_n(q)$ (resp. $\mathrm{GU}_n(q)$) and $G = \mathrm{SL}_n(q)$ (resp. $G = \mathrm{SU}_n(q)$). The unitriangular basic set that we obtain for G is explicitly built from a unitriangular basic set for $\tilde{G}$. We develop in the first part, in a general setting, a condition that ensures the existence of a stable unitriangular basic set for G provided that we already have one for $\tilde{G}$. The second part recalls some known facts about the character theory of general linear and unitary groups, and the last part is devoted to proving Theorem A.

2. Stable unitriangular basic set for a normal subgroup

2.1. Decomposition numbers, unitriangularity and stability under group action

Our basic reference for the representation theory of finite groups is [20]. Let H be a finite group and ℓ be a prime number dividing the order of H. The set of ordinary irreducible characters of H will be denoted by $\mathrm{Irr}(H)$ and its elements will be referred to simply as irreducible characters. The set of (irreducible ℓ-) Brauer characters of H will be denoted by $\mathrm{IBr}_\ell(H)$ or simply $\mathrm{IBr}(H)$. The set $\mathrm{Irr}(H)$ forms a basis of the space of class functions on H, which is endowed with its usual scalar product denoted by $\langle -, - \rangle$. The set $\mathrm{IBr}(H)$ is a basis of the subspace of class functions on H vanishing outside the