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Characterization of order types of pointwise linearly ordered families of Baire class 1 functions

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ABSTRACT

In the 1970s M. Laczkovich posed the following problem: Let $\mathcal{B}_1(X)$ denote the set of Baire class 1 functions defined on an uncountable Polish space X equipped with the pointwise ordering.

> Characterize the order types of the linearly ordered subsets of $\mathcal{B}_1(X)$.

The main result of the present paper is a complete solution to this problem.
We prove that a linear order is isomorphic to a linearly ordered family of Baire class 1 functions iff it is isomorphic to a subset of the following linear order that we call $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, where $[0,1]^{<\omega_1}_{\searrow 0}$ is the set of strictly decreasing transfinite sequences of reals in $[0,1]$ with last element 0, and $<_{altlex}$, the so called *alternating lexicographical ordering*, is defined as follows: if $(x_\alpha)_{\alpha\le\xi}, (x'_\alpha)_{\alpha\le\xi'} \in [0,1]^{<\omega_1}_{\searrow 0}$ are distinct, and δ is the minimal ordinal where the two sequences differ then we say that

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$(x_\alpha)_{\alpha\le\xi} <_{altlex} (x'_\alpha)_{\alpha\le\xi'} \iff$ (δ is even and $x_\delta < x'_\delta$) or (δ is odd and $x_\delta > x'_\delta$).

Using this characterization we easily reprove all the known results and answer all the known open questions of the topic.

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Contents

1. Introduction . . . 560
2. Preliminaries . . . 563
3. The main result . . . 565
3.1. $\mathcal{B}_1(X) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$ . . . 565
3.2. $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow \mathcal{B}_1(X)$ . . . 568
3.3. The main theorem . . . 574
4. New proofs of known theorems . . . 575
4.1. Kuratowski's theorem . . . 575
4.2. Komjáth's theorem . . . 576
4.3. Linearly ordered sets of cardinality $< \mathfrak{c}$ and Martin's Axiom . . . 578
5. New results . . . 581
5.1. Countable products and gluing . . . 581
5.2. Completion . . . 585
6. Proof of Proposition 3.5 . . . 591
7. Open problems . . . 595
Acknowledgments . . . 596
References . . . 596

1. Introduction

Let $\mathcal{F}(X)$ be a class of real valued functions defined on a Polish space X, e.g. $C(X)$, the set of continuous functions. The natural partial ordering on this space is the pointwise ordering $<_p$, that is, we say that $f <_p g$ if for every $x \in X$ we have $f(x) \le g(x)$ and there exists at least one x such that $f(x) < g(x)$. If we would like to understand the structure of this partially ordered set (poset), the first step is to describe its linearly ordered subsets.

For example, if $X = [0,1]$ and $\mathcal{F}(X) = \mathcal{C}([0,1])$ then it is a well known result that the possible order types of the linearly ordered subsets of $\mathcal{C}([0,1])$ are the real order types (that is, the order types of the subsets of the reals). Indeed, a real order type is clearly representable by constant functions, and if $\mathcal{L} \subset \mathcal{C}([0,1])$ is a linearly ordered family of continuous functions then (by continuity) $f \mapsto \int_0^1 f$ is a *strictly* monotone map of $\mathcal{L}$ into the reals.

The next natural class to look at is the class of Lebesgue measurable functions. However, it is not hard to check that the assumption of measurability is rather meaningless here. Indeed, if $\mathcal{L}$ is a linearly ordered family of *arbitrary* real functions and $\varphi : \mathbb{R} \to \mathbb{R}$

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