Advances in Mathematics 307 (2017) 727–753

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The Novikov conjecture for algebraic K-theory of the group algebra over the ring of Schatten class operators

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ARTICLE INFO

Article history:
Received 8 November 2016
Accepted 21 November 2016
Available online xxxx
Communicated by the Managing Editors

Keywords:
Novikov conjecture
Algebraic K-theory
Group algebra
Schatten class operators

ABSTRACT

In this paper, we prove the algebraic K-theory Novikov conjecture for group algebras over the ring of Schatten class operators. The main technical tool in the proof is an explicit construction of the Connes–Chern character.

1. Introduction

Let Γ be a group and R be an H-unital ring. Let $R\Gamma$ be the group algebra of the group Γ over the ring R. The isomorphism conjecture of Farrell–Jones states that the following assembly map is an isomorphism:

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[1] The author is partially supported by NSF 1362772 and CNSF 11420101001.

http://dx.doi.org/10.1016/j.aim.2016.11.027
0001-8708/

$$A : H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(R)^{-\infty}) \longrightarrow K_n(R\Gamma),$$

where $\mathcal{VCY}$ is the family of virtually cyclic subgroups of Γ, $E_{\mathcal{VCY}}(\Gamma)$ is the universal Γ-space with isotropy in $\mathcal{VCY}$, $H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(R)^{-\infty})$ is a generalized Γ-equivariant homology theory associated to the non-connective algebraic K-theory spectrum $\mathbb{K}(R)^{-\infty}$, and $K_n(R\Gamma)$ is the algebraic K-theory of $R\Gamma$.

The isomorphism conjecture provides an algorithm for computing the algebraic K-theory of $R\Gamma$ in terms of the algebraic K-theory of R. This conjecture was introduced in [18] for $R = \mathbb{Z}$ and for unital rings R in [1]. When R is H-unital, the isomorphism conjecture follows from the unital case by using the excision theorem in algebraic K-theory [32]. The algebraic K-theory isomorphism conjecture goes back to [21]. There are analogous conjectures in L-theory [26,27] and C^*-algebra K-theory [4]. Important cases of the isomorphism conjecture have been verified in [18,19,2].

The algebraic K-theoretic Novikov conjecture states that the assembly map:

$$H_n(B\Gamma, \mathbb{K}(R)^{-\infty}) \longrightarrow K_n(R\Gamma),$$

is rationally injective, where $B\Gamma$ is the classifying space of the group Γ. The algebraic K-theoretic Novikov conjecture follows from the (rational) injectivity part of the isomorphism conjecture. By a remarkable theorem of Bökstedt–Hsiang–Madsen [6], the algebraic K-theoretic Novikov conjecture holds for $R = \mathbb{Z}$ if the homology groups of Γ are finitely generated.

The main purpose of this paper is to prove the (rational) injectivity part of the algebraic K-theory isomorphism conjecture for group algebras over the ring of Schatten class operators. As a consequence, we obtain the algebraic K-theory Novikov conjecture for group algebras over the ring of Schatten class operators. The motivation for considering group algebras over the ring of Schatten class operators comes from the deep work of Connes–Moscovici on higher index theory of elliptic operators and its applications to the Novikov conjecture [11]. In Connes–Moscovici's higher index theory, the K-theory of the group algebra over the ring of Schatten class operators serves as the receptacle for the higher index of an elliptic operator.

For the convenience of readers we recall that, for any $p \geq 1$, an operator T on an infinite dimensional and separable Hilbert space H is said to be Schatten p-class if $tr((T^*T)^{p/2}) < \infty$, where tr is the standard trace defined by $tr(P) = \sum_n < Pe_n, e_n >$ for any bounded operator P acting on H and an orthonormal basis $\{e_n\}_n$ of H ($tr(P)$ is independent of the choice of the orthonormal basis). Let $\mathcal{S}_p$ be the ring of all Schatten p-class operators on an infinite dimensional and separable Hilbert space. We define the ring $\mathcal{S}$ of all Schatten class operators to be $\cup_{p\geq 1}\mathcal{S}_p$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $\mathcal{S}$ be the ring of all Schatten class operators on an infinite dimensional and separable Hilbert space. The assembly map*