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A local theory for operator tuples in the Cowen–Douglas class

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ABSTRACT

We present a local theory for a commuting m-tuple $\mathbf{S} = (S_1, S_2, \cdots, S_m)$ of Hilbert space operators lying in the Cowen–Douglas class. By representing $\mathbf{S}$ on a Hilbert module $\mathcal{M}$ consisting of vector-valued holomorphic functions over $\mathbb{C}^m$, we identify and study the localization of $\mathbf{S}$ on an analytic hyper-surface in $\mathbb{C}^m$. We completely determine unitary equivalence of the localization and relate it to geometric invariants of the Hermitian holomorphic vector bundle associated to $\mathbf{S}$. It turns out that the localization coincides with an important class of quotient Hilbert modules, and our result concludes its classification problem in full generality.

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1. Introduction

Coordinate multiplications on a holomorphic function space of several complex variables provide a basic model in multivariate operator theory. For fixed positive integers m and n, let $\mathcal{M}$ be a Hilbert space consisting of $\mathbb{C}^n$-valued holomorphic functions over a connected bounded open subset Ω in $\mathbb{C}^m$ such that for any $1 \leq i \leq m$, the operator

$$T_i : f \mapsto z_i f, f \in \mathcal{M}$$

of multiplication by the i^{th} coordinate function z_i is bounded on $\mathcal{M}$. We assume the following standard properties of $\mathcal{M}$ throughout this paper:

(a) For any $\mathbf{z} = (z_1, \cdots, z_m) \in \Omega$, the evaluation map $e_{\mathbf{z}}$: $f \mapsto f(\mathbf{z}), f \in \mathcal{M}$, is bounded from $\mathcal{M}$ *onto* $\mathbb{C}^n$,

(b) $\mathcal{M}$ has the Gleason property, i.e., for any $\mathbf{z} \in \Omega$ and $f \in \mathcal{M}$, $f(\mathbf{z}) = \mathbf{0}$ if and only if $f = (T_1 - z_1)f_1 + \cdots + (T_m - z_m)f_m$ for some $f_1, \cdots, f_m$ in $\mathcal{M}$.

In the scalar case $n = 1$, examples include well-studied function spaces such as the Hardy or Bergman spaces over the unit ball or polydisc in $\mathbb{C}^m$, and trivial vector-valued example can be obtained by taking the direct sum of n-copies of a scalar one. In general, the structure of a vector-valued function space can be quite complicated, and one can view $\mathcal{M}$ as a *Hilbert module* ([8]) in the sense that $\mathcal{M}$ as a Hilbert space admits a natural module structure over the polynomial ring $\mathbb{C}[z_1, \cdots, z_m]$ with respect to the obvious multiplication action.

A basic problem in operator theory is to determine when two operators or operator tuples are unitarily equivalent. In this paper we will investigate the problem for the m-tuple $(T_1, \cdots, T_m)$ from a local perspective, which is motivated by the seminal work of Cowen and the second author [3] on a complex geometric approach to operator theory, as well as some recent developments on classification of Hilbert modules.

Definition 1. Given a positive integer n and a bounded domain Ω in $\mathbb{C}^m$, a commuting m-tuple of operators $\mathbf{S} = (S_1, \cdots, S_m)$ acting on a Hilbert space H belongs to the Cowen–Douglas class, denoted by $\mathcal{B}_n^m(\Omega)$, if the following holds:

(i) The space $\{((S_1 - z_1)h, \cdots, (S_m - z_m)h), h \in H\}$ is a closed subspace in $H \oplus \cdots \oplus H$ (m copies of H) for every $\mathbf{z} = (z_1, \cdots, z_m) \in \Omega$;

(ii) $\vee_{\mathbf{z} \in \Omega} \cap_{i=1}^m \ker(S_i - z_i) = H$ (here $\vee$ denotes the closed linear span); and

(iii) $\dim \cap_{i=1}^m \ker(S_i - z_i) = n$ for every $\mathbf{z} \in \Omega$.

The $\mathcal{B}_n^1(\Omega)$ class was originally introduced in [3] for a single operator. Curto and Salinas [5] extended it to operator tuples and showed that the $\mathcal{B}_n^m(\Omega)$ class can actually be modeled by adjoints of coordinate multiplication on a Hilbert module as we just mentioned in the beginning.