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On the distance sets of Ahlfors–David regular sets

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ABSTRACT

I prove that if $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors–David regular set with $s \geq 1$, then

$$\dim_{\mathrm{p}} D(K) = 1,$$

where $D(K) := \{|x-y| : x, y \in K\}$ is the distance set of K, and $\dim_{\mathrm{p}}$ stands for packing dimension.
The same proof strategy applies to other problems of similar nature. For instance, one can show that if $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors–David regular set with $s \geq 1$, then there exists a point $x_0 \in K$ such that $\dim_{\mathrm{p}} K \cdot (K - x_0) = 1$.

1. Introduction

Given a planar set K, the *distance set problem* asks for a relationship between the size of K, and the size of the distance set

$$D(K) := \{|x-y| : x, y \in K\}.$$

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For finite sets K, the problem is due to P. Erdős from 1946, and the *Erdős distance conjecture* states that the cardinality of $D(K)$ should satisfy $|D(K)| \gtrsim |P|/\sqrt{\log |P|}$. L. Guth and N. Katz [7] nearly resolved the question in 2011 by showing that $|D(K)| \gtrsim |P|/\log |P|$.

The "continuous" version of the distance set problem was proposed by K. Falconer [4] in 1985. The *Falconer distance conjecture* claims that if $K \subset \mathbb{R}^2$ is a Borel set with $\dim K > 1$, then $D(K)$ has positive length. As far as I know, the current records for general sets in this setting are the following theorems of T. Wolff [13] from 1999 and J. Bourgain [3] from 2003:

Theorem 1.1 *(Wolff). If $K \subset \mathbb{R}^2$ is Borel with $\dim K > 4/3$, then $D(K)$ has positive length.*

Theorem 1.2 *(Bourgain). If $K \subset \mathbb{R}^2$ is Borel with $\dim K \geq 1$, then $\dim_{\mathrm{H}} D(K) \geq 1/2 + \epsilon$ for some (small) absolute constant $\epsilon > 0$.*

In Bourgain's result, $\dim_{\mathrm{H}}$ stands for Hausdorff dimension. Recent years have witnessed considerable interest in trying to prove Falconer's conjecture – or at least improve significantly upon Theorems 1.1 and 1.2 – for special classes of sets. Bárány [1] and the author [12] considered self-similar sets in $\mathbb{R}^2$ and $\mathbb{R}^3$. Ferguson, Fraser and Sahlsten [5] studied some classes of self-affine sets in the plane. Most recently, Fraser and Pollicott [6] investigated planar self-conformal sets. I will not state the precise contributions of these papers individually, but each contains a result of the following kind: if K is a set in the special class under consideration, with positive linear measure, then $\dim_{\mathrm{H}} D(K) = 1$. For planar self-similar sets K, Bárány's result [1] is even stronger: $\dim_{\mathrm{H}} K = 1$ already implies $\dim_{\mathrm{H}} D(K) = 1$. In the present paper, I consider the class of Ahlfors–David regular sets, and the main result is the following:

Theorem 1.3. *Assume that $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors–David-regular set with $s \geq 1$. Then*

$$\dim_{\mathrm{p}} D(K) = 1,$$

where $\dim_{\mathrm{p}}$ stands for packing dimension, see Definition 1.6 below.

The precise definition of Ahlfors–David regular sets is Definition 1.7 below; for instance, self-similar sets satisfying the open set condition are Ahlfors–David regular. However, as the following construction shows, Ahlfors–David regular sets are not, in general, associated with an obvious dynamical system:

Example 1.4. Fix two integers $0 < m \leq n^2$ and let $\mathcal{Q}_0 := \{[0,1]^2\}$. Next, assume that $j \geq 0$, and $\mathcal{Q}_j$ is a collection of interior-disjoint closed squares of side-length n^{-j}. Fix $Q \in \mathcal{Q}_j$. Divide Q into n^2 closed squares of side-length n^{-j-1}, specify (any) m of them,