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Moduli space for generic unfolded differential linear systems ☆

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A B S T R A C T

In this paper, we identify the moduli space for germs of generic unfoldings of nonresonant linear differential systems with an irregular singularity of Poincaré rank k at the origin, under analytic equivalence. The modulus of a given family was determined in [10]: it comprises a formal part depending analytically on the parameters, and an analytic part given by unfoldings of the Stokes matrices. These unfoldings are given on "Douady–Sentenac" (DS) domains in the parameter space covering the generic values of the parameters corresponding to Fuchsian singular points. Here we identify exactly which moduli can be realized. A necessary condition on the analytic part, called compatibility condition, is saying that the unfoldings define the same monodromy group (up to conjugacy) for the different presentations of the modulus on the intersections of DS domains. With the additional requirement that the corresponding cocycle is trivial and good limit behavior at some boundary points of the DS domains, this condition becomes sufficient. In particular we show that any modulus can be realized by a k-parameter family of systems of rational linear differential equations over $\mathbb{CP}^1$ with $k+1$, $k+2$ or $k+3$ singular points (with multiplicities). Under the generic condition of irreducibility, there are precisely $k+2$ singular

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points which are Fuchsian as soon as simple. This in turn implies that any unfolding of an irregular singularity of Poincaré rank k is analytically equivalent to a rational system of the form $y' = \frac{A(x)}{p_\epsilon(x)} \cdot y$, with $A(x)$ polynomial of degree at most k and $p_\epsilon(x)$ is the generic unfolding of the polynomial x^{k+1}.

1. Introduction

The local classification in the complex domain of germs of systems of linear differential equations, with a pole at the origin,

$$y' = \frac{A(x)}{x^{k+1}} \cdot y,$$

exhibits a qualitative shift as one goes from $k = 0$ to $k > 0$. In both cases, one can first go to a normal form by a formal gauge transformation $g(x)$, that is a power series in x. Let us suppose for simplicity that the system is nonresonant i.e. the leading term $A(0)$ is diagonal, with eigenvalues which are distinct (for $k = 0$, one would ask that they be distinct modulo the integers). One can perform a formal normalization to have $A(x)$ diagonal, and a polynomial of order k. If we then proceed to the analytic classification, one finds that for $k = 0$, the formal classification is the same as the analytic classification, in the absence of resonance. For $k > 0$, the situation is very different. The formal gauge transformation does not in general converge, and one only has analytic solutions on $2k$ sectors around the origin, with constant matrices (Stokes matrices) relating the solutions as one goes from sector to sector. If we further assume that $A(0) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, and that we have permuted the coordinates of y and rotated x to $e^{i\theta}x$ so that

$$\mathrm{Re}(\lambda_1) > \cdots > \mathrm{Re}(\lambda_n), \tag{1.1}$$

then the Stokes matrices alternate between upper triangular and lower triangular as one goes from sector to sector. Once one has fixed the formal normal form, the Stokes matrices provide complete invariants. While these can be thought of as generalized monodromies (e.g., [18]), the passage from the irregular case ($k > 0$) to the regular case $k = 0$ is not immediate, since the monodromies for $k = 0$ have no limit at the confluence. This passage however is a natural one to consider, in particular when unfolding a system with an irregular singularity. Doing so sheds new light on the meaning of the Stokes matrices, and this has been studied in particular in [19,8,15,10].

Indeed, one has a deformation from one to the other. Let $p_\epsilon(x)$, $\epsilon \in \mathbb{C}^k$ be the generic deformation of x^{k+1} as a polynomial of degree $k + 1$:

$$p_\epsilon(x) = x^{k+1} + \epsilon_{k-1}x^{k-1} + \cdots + \epsilon_1 x + \epsilon_0, \tag{1.2}$$

and then consider a deformed system