Advances in Mathematics 309 (2017) 209–237

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Equivariant Moore spaces and the Dade group

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ARTICLE INFO

Article history:
Received 29 June 2016
Accepted 17 January 2017
Communicated by Henning Krause

MSC:
primary 57S17
secondary 20C20

Keywords:
Equivariant Moore spaces
Dade group
Borel–Smith functions
Orbit category
Biset functors

ABSTRACT

Let G be a finite p-group and k be a field of characteristic p. A topological space X is called an n-Moore space if its reduced homology is nonzero only in dimension n. We call a G-CW-complex X an $\underline{n}$-Moore G-space over k if for every subgroup H of G, the fixed point set X^H is an $\underline{n}(H)$-Moore space with coefficients in k, where $\underline{n}(H)$ is a function of H. We show that if X is a finite $\underline{n}$-Moore G-space, then the reduced homology module of X is an endo-permutation kG-module generated by relative syzygies. A kG-module M is an endo-permutation module if $\mathrm{End}_k(M) = M \otimes_k M^*$ is a permutation kG-module. We consider the Grothendieck group of finite Moore G-spaces $\mathcal{M}(G)$, with addition given by the join operation, and relate this group to the Dade group generated by relative syzygies.

1. Introduction and statement of results

Let G be a finite group and M be a $\mathbb{Z}G$-module. A G-CW-complex X is called a *Moore G-space* of type (M, n) if the reduced homology group $\widetilde{H}_i(X;\mathbb{Z})$ is zero whenever $i \neq n$ and $\widetilde{H}_n(X;\mathbb{Z}) \cong M$ as $\mathbb{Z}G$-modules. One of the classical problems in algebraic topology, due to Steenrod, asks whether every $\mathbb{Z}G$-module is realizable as the homology module of a Moore G-space. G. Carlsson [10] constructed counterexamples of non-realizable modules

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http://dx.doi.org/10.1016/j.aim.2017.01.017

for finite groups that include $\mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup for some prime p. The question of finding a good algebraic characterization of realizable $\mathbb{Z}G$-modules is still an open problem (see [21] and [4]).

In this paper we consider Moore G-spaces whose fixed point subspaces are also Moore spaces. Let R be a commutative ring of coefficients and let $\underline{n} : \mathrm{Sub}(G) \to \mathbb{Z}$ denote a function from subgroups of G to integers, which is constant on the conjugacy classes of subgroups. Such functions are often called super class functions.

Definition 1.1. A G-CW-complex X is called an *$\underline{n}$-Moore G-space over R* if for every $H \leq G$, the reduced homology group $\widetilde{H}_i(X^H; R)$ is zero for all $i \neq \underline{n}(H)$.

When $\underline{n}$ is the constant function with value n for all $H \leq G$, the homology at dimension n can be considered as a module over the orbit category $\mathrm{Or}\, G$. If $\underline{\widetilde{H}}_n(X^?; R) \cong \underline{M}$ as a module over the orbit category, then X is called a Moore G-space of type $(\underline{M}, n)$. When $R = \mathbb{Q}$ and X^H is simply-connected for all $H \leq G$, the space X is called a rational Moore G-space. Rational Moore G-spaces are studied extensively in equivariant homotopy theory and many interesting results are obtained on homotopy types of these spaces (see [16] and [11]).

In this paper, we allow $\underline{n}$ to be an arbitrary super class function and take the coefficient ring R as a field k of characteristic p. We define the group of finite Moore G-spaces over k and relate this group to the Dade group, the group of endo-permutation modules. Since the appropriate definition of a Dade group for a finite group is not clear yet, we restrict ourselves to the situation where G is a p-group, although the results have obvious consequences for finite groups via restriction to a Sylow p-subgroup.

Let G be a finite group and k be a field of characteristic p. Throughout we assume all kG-modules are finitely generated. A kG-module M is called an *endo-permutation kG-module* if $\mathrm{End}_k(M) = M \otimes_k M^*$ is isomorphic to a permutation kG-module when regarded as a kG-module with diagonal G-action $(gf)(m) = gf(g^{-1}m)$. A G-CW-complex is called *finite* if it has finitely many cells. The main result of the paper is the following:

Theorem 1.2. *Let G be a finite p-group, and k be a field of characteristic p. If X is a finite $\underline{n}$-Moore G-space over k, then the reduced homology module $\widetilde{H}_n(X, k)$ in dimension $n = \underline{n}(1)$ is an endo-permutation kG-module generated by relative syzygies.*

This theorem is proved in Sections 3 and 4. We first prove it for tight Moore G-spaces (Proposition 3.8) and then extend it to the general case. An $\underline{n}$-Moore space X is said to be *tight* if the topological dimension of X^H is equal to $\underline{n}(H)$ for every $H \leq G$. For tight Moore G-spaces, we give an explicit formula that expresses the equivalence class of the homology group $\widetilde{H}_n(X, k)$ in terms of relative syzygies (see Proposition 3.8). This formula plays a key role for relating the group of Moore G-spaces to the group of Borel–Smith functions and to the Dade group. We now introduce these groups and the homomorphisms between them.