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On the scattering operators for ACHE metrics of Bergman type on strictly pseudoconvex domains

Fang Wang [1]

Shanghai Jiao Tong University, 800 Dongchuan Rd, Shanghai 200240, China

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ABSTRACT

The scattering operators associated to an ACHE metric of Bergman type on a strictly pseudoconvex domain are a one-parameter family of CR-conformally invariant pseudo-differential operators of Heisenberg class with respect to the induced CR structure on the boundary. In this paper, we mainly show that if the boundary Webster scalar curvature is positive, then for $\gamma \in (0, 1)$ the renormalised scattering operator $P_{2\gamma}$ has positive spectrum and satisfies the maximum principal; moreover, the fractional curvature $Q_{2\gamma}$ is also positive. This is parallel to the result of Guillarmou–Qing [16] for the real case. We also give two energy extension formulae for $P_{2\gamma}$, which are parallel to the energy extension given by Chang–Case [2] for the real case.

1. Introduction

The scattering operators associated to the Laplacian operator for a *real asymptotically hyperbolic* manifold have been extensively studied, see for example [28,12,20,16,2] and the references cited there. The purpose of this paper is to extend some results in [16] and

E-mail address: fangwang1984@sjtu.edu.cn.

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[2] to *asymptotically complex hyperbolic* (ACH) manifolds. Similarly as in [16] and [2], the author here is particularly interested in (approximate) *asymptotically complex hyperbolic Einstein* (ACHE) manifolds with infinity of positive CR-Yamabe type.

Before discussing the asymptotically complex hyperbolic manifolds, let us first recall some results for real asymptotically hyperbolic manifolds, which should help us to understand the complex case. Suppose X is a manifold with boundary of dimension $n+1$ and ρ is a smooth boundary defining function. Let g be a smooth metric in the interior $\mathring{X}$ such that $\bar{g} = \rho^2 g$ is smooth and nondegenerate up to the boundary, satisfying $|d\rho|^2_{\bar{g}} \to 1$ when $\rho \to 0$. Then in a collar neighbourhood of M, denoted by $[0, \epsilon)_\rho \times M$,

$$\bar{g} = d\rho^2 + g_\rho$$

where g_ρ is a smooth-one parameter family of Riemannian metric on the boundary M. Then g is asymptotically hyperbolic in the sense that all the sectional curvature has limit -1 when approaching the boundary. The metric g induces a conformal class $[g_0]$ on the boundary by choosing different boundary defining functions. A standard example is the ball model of real hyperbolic space $\mathbb{H}^{n+1}$, i.e. the unit ball $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}_z$ equipped with metric

$$h = 4(1 - |z|^2)^{-2} dz^2.$$

The spectrum and resolvent for the Laplacian operator $\triangle_g$ for (X, g) were studied by Mazzeo–Melrose [27], Mazzeo [26] and Guillarmou [14]. The spectrum of $\triangle_g$ consists of two disjoint parts, the absolute continuous spectrum $\sigma_{ac}(\triangle_g)$ and the pure point spectrum $\sigma_{pp}(\triangle_g)$ (i.e. L^2-eigenvalues). More explicitly,

$$\sigma_{ac}(\triangle_g) = \left[n^2/4, \infty\right), \quad \sigma_{pp}(\triangle_g) \subset \left(0, n^2/4\right).$$

The resolvent $R(\lambda) = (\triangle_g - \lambda(n-\lambda))^{-1}$ is a bounded operator on $L^2(X, \mathrm{dvol}_g)$ for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \frac{n}{2}$, $\lambda(n-\lambda) \notin \sigma_{pp}(\triangle_g)$, which has finite meromorphic extension to $\mathbb{C}\backslash\{\frac{n-1}{2} - K - \mathbb{N}_0\}$, where $2K$ is the order up to which the Taylor expansion of g_ρ is even. For example, if g_ρ has complete even Taylor expansion at $\rho = 0$, then $R(\lambda)$ has finite meromorphic extension to entire $\mathbb{C}$. If g is Einstein (i.e. g is Poincaré–Einstein), then for n odd, the Taylor expansion of g_ρ is even up to order $n-1$ while choosing ρ to be the geodesic normal defining function. In this case, $R(\lambda)$ has meromorphic extension to $\mathbb{C}\backslash(-\mathbb{N}_0)$. For n even, a Poincaré–Einstein metric g has logarithmic terms in the asymptotic expansion of g_ρ. This makes $\bar{g} = \rho^2 g$ not smooth up to the boundary. However, the analysis of spectrum and resolvent above is still valid except that the mapping property of $R(\lambda)$ changes a bit. See [4] for the asymptotic expansion of Poincaré–Einstein metric and [15] for more details on meromorphic extension of the resolvent. In particular, Lee [22] showed that for Poincaré–Einstein metric, if the conformal infinity is of nonnegative Yamabe type, then there is no L^2-eigenvalue and hence $\sigma(\triangle_g) = \sigma_{ac}(\triangle_g) = [n^2/4, \infty)$.