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On the Andrews–Zagier asymptotics for partitions without sequences

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we establish asymptotics of radial limits for certain functions of Wright. These functions appear in bootstrap percolation and the generating function for partitions without sequences of k consecutive part sizes. We specifically establish asymptotics numerically obtained by Zagier in the case $k = 3$.

1. Introduction and statement of results

Holroyd, Liggett, and Romik [8] introduced the following probability models: Let $0 < s < 1$ and $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be independent events with probabilities

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$$\mathbf{P}_s(\mathcal{C}_n) := 1 - e^{-ns}$$

under a certain probability measure $\mathbf{P}_s$. Let A_k be the event

$$A_k := \bigcap_{j=1}^{\infty} \left(\mathcal{C}_j \cup \mathcal{C}_{j+1} \cup \cdots \cup \mathcal{C}_{j+k-1}\right)$$

that there is no sequence of k consecutive $\mathcal{C}_j$ that do not occur. With $q := e^{-s}$ throughout the remainder of the paper, set

$$g_k(q) := \mathbf{P}_s(A_k).$$

To solve a problem in bootstrap percolation, Holroyd, Liggett, and Romik established an asymptotic for $\log(g_k(e^{-s}))$.

Interestingly, the above described probability model also appears in the study of integer partitions [4,8]. In particular,

$$G_k(q) = g_k(q) \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

is the generating function for the number of integer partitions without k consecutive part sizes. Partitions without 2 consecutive parts have a celebrated history in relation to the famous Rogers–Ramanujan identities. See MacMahon's book [10] or the works of Andrews [1–3] for more about such partitions.

Andrews [3] found that the key to understanding the function if $k = 2$ lies in Ramanujan's mock theta function

$$\chi(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}\left(1 - q^j + q^{2j}\right)}.$$

Namely, he proved that

$$g_2(q) = \chi(q) \prod_{n=1}^{\infty} \frac{\left(1+q^{3n}\right)}{\left(1-q^n\right)\left(1-q^{2n}\right)}.$$

From this, an asymptotic expansion for $g_2(e^{-s})$ may be obtained (see [5]). Using additional q-series identities if $k > 2$, Andrews made the following conjecture.

Conjecture 1.1 *(Andrews [3]). For each $k \geq 2$, there exists a positive constant C_k such that, as $s \to 0$,*

$$g_k\left(e^{-s}\right) \sim C_k s^{-\frac{1}{2}} \exp\left(-\frac{\pi^2}{3k(k+1)s}\right).$$