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Some congruences modulo 2 and 5 for bipartition with 5-core

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Abstract. We find some congruences modulo 2 and 5 for the number of bipartitions with 5-core for a positive integer n in the spirit of Ramanujan.

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1. Introduction

A bipartition of a positive integer n is a pair of partitions (λ, μ) such that the sum of all of the parts is n. A bipartition with t-core is a pair of partitions (λ, μ) such that λ and μ are both t-cores. If $A_t(n)$ denotes the number of bipartitions with t-core of n, then $A_t(n)$ is defined by

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{(q^t;q^t)_\infty^{2t}}{(q;q)_\infty^2}, \tag{1.1}$$

where $(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^n)$. We note the following well known congruence property which can be proved by using binomial theorem: For any prime p and positive integer k,

$$(q^k;q^k)_\infty^p \equiv (q^{pk};q^{pk})_\infty \pmod{p}. \tag{1.2}$$

The function $A_t(n)$ defined in (1.1) have been studied by many mathematicians. Lin [8] discovered some interesting congruences modulo 4, 5, 7, and 8 for $A_3(n)$. Yao [10] established several infinite families of congruences modulo 3 and 9 for $A_9(n)$. Xia [9]

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established several infinite families of congruences modulo 4, 8 and $\frac{4^k-1}{3}$ $(k \geq 2)$ for $A_3(n)$ and also generalized some results due to Lin and Yao. Baruah and Nath [1] also proved some results on $A_3(n)$.

In this paper, we are concerned with the function $A_5(n)$ which denotes the number of bipartition with 5-core of n and is given by

$$\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}. \tag{1.3}$$

In Section 3, we find some congruences modulo 2 and 5 for $A_5(n)$ in the spirit of Ramanujan. Section 2 is devoted to record some preliminary results.

2. Preliminaries

Ramanujan's general theta-function $f(a,b)$ [3, p. 35, Entry 19] is defined by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1. \tag{2.1}$$

Lemma 2.1 *([4, Theorem 2.2]). For any prime $p \geq 5$, we have*

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty, \tag{2.2}$$

where $\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6}, & if\ p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6}, & if\ p \equiv -1 \pmod 6. \end{cases}$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p-1}{2}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

Lemma 2.2 *([7, Theorem 1]). We have*

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^{20};q^{20})_\infty}.$$

Lemma 2.3 *([6]). We have*

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}(F^{-4}(q^5) + qF^{-3}(q^5)$$
$$+ 2q^2F^{-2}(q^5) + 3q^3F^{-1}(q^5) + 5q^4 - 3q^5F(q^5)$$
$$+ 2q^6F^2(q^5) - q^7F^3(q^5) + q^8F^4(q^5)),$$

where $F(q) := q^{-1/5}R(q)$ and $R(q)$ is Rogers-Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{q}{+1} \, \frac{q^2}{+1} \, \frac{q^3}{+1} \, {}_{+\cdots}, \qquad |q| < 1.$$