ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 23 (2017) 141–147

Some explicit and recursive formulas of the large and little Schröder numbers

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Received 4 April 2016; accepted 13 June 2016
Available online 21 June 2016

Abstract. In the paper, the authors analytically find some explicit formulas and recursive formulas for the large and little Schröder numbers.

2010 Mathematics Subject Classification: primary 11Y55; secondary 05A15; 11B83; 11R33; 26A48; 30E20; 44A10; 44A15

Keywords: Explicit formula; Recursive formula; Schröder number; Generating function

1. INTRODUCTION

In combinatorics and number theory, there are two kinds of Schröder numbers, the large Schröder numbers S_n and the little Schröder numbers s_n. They are named after the German mathematician Ernst Schröder.

A large Schröder number S_n describes the number of paths from the southwest corner $(0,0)$ of an $n\times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest–northeast diagonal. The first eleven large Schröder

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.06.002

numbers S_n for $0 \le n \le 10$ are

$$1, \quad 2, \quad 6, \quad 22, \quad 90, \quad 394, \quad 1806, \quad 8558, \quad 41586, \quad 206098, \quad 1037718.$$

In [1, Theorem 8.5.7], it was proved that the large Schröder numbers S_n have the generating function

$$G(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x} = \sum_{n=0}^{\infty} S_n x^n, \tag{1}$$

which can also be rearranged as

$$\mathcal{G}(x) = G(-x) = \frac{\sqrt{x^2 + 6x + 1} - 1 - x}{2x} = \sum_{n=0}^{\infty} (-1)^n S_n x^n. \tag{2}$$

The little Schröder numbers s_n form an integer sequence that can be used to count the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. The first eleven little Schröder numbers s_n for $1 \le n \le 11$ are

$$1, \quad 1, \quad 3, \quad 11, \quad 45, \quad 197, \quad 903, \quad 4279, \quad 20793, \quad 103049, \quad 518859.$$

They are also called the small Schröder numbers, the Schröder–Hipparchus numbers, or the Schröder numbers, after Ernst Schröder and the ancient Greek mathematician Hipparchus who appears from evidence in Plutarch to have known of these numbers. They are also called the super-Catalan numbers, after Eugéne Charles Catalan, but different from a generalization of the Catalan numbers [2,10]. In [1, Theorem 8.5.6], it was proved that the little Schröder numbers s_n have the generating function

$$g(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4} = \sum_{n=1}^{\infty} s_n x^n. \tag{3}$$

For more information on the large Schröder numbers S_n and the little Schröder numbers s_n, please refer to [1,7–9] and plenty of references therein.

Comparing (1) with (3), we can reveal

$$\sqrt{x^2 - 6x + 1} = 1 + x - 4\sum_{n=1}^{\infty} s_n x^n = 1 - x - 2\sum_{n=0}^{\infty} S_n x^{n+1},$$

that is,

$$1 - 2\sum_{n=1}^{\infty} s_n x^{n-1} = 1 - 2\sum_{n=0}^{\infty} s_{n+1} x^n = -\sum_{n=0}^{\infty} S_n x^n.$$

Accordingly, we acquire

$$S_n = 2s_{n+1}, \quad n \in \mathbb{N}. \tag{4}$$

See also [1, Corollary 8.5.8]. This relation tells us that it is sufficient to analytically study the large Schröder numbers S_n.

Recently, in the paper [3] and the preprints [4–6], some new conclusions, including several explicit formulas, integral representations, and some properties such as the convexity,