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On the equation $V_n = wx^2 \mp 1$

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Abstract. Let $P \geq 3$ be an integer and (V_n) denote Lucas sequence of the second kind defined by $V_0 = 2, V_1 = P$, and $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. In this study, when P is odd and $w \in \{10, 14, 15, 21, 30, 35, 42, 70, 210\}$, we solved the equation $V_n = wx^2 \mp 1$. We showed that only V_1 can be of the form $wx^2 + 1$ and only V_1 or V_2 can be of the form $wx^2 - 1$.

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Keywords: Lucas sequence of the second kind; Congruences; Diophantine equations

1. INTRODUCTION

Let $P \geq 3$ be an integer and (V_n) denote Lucas sequence of the second kind defined by $V_0 = 2, V_1 = P$, and $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$.

In [1], the authors showed that when $a \neq 0$ and $b \neq \pm 2$, the equation $V_n = ax^2 + b$ has only a finite number of solutions n. In [5], Keskin solved the equations $V_n = wx^2 + 1$ and $V_n = wx^2 - 1$ for $w = 1, 2, 3, 6$ when P is odd. In [4], when P is odd, Karaatlı and Keskin solved the equations $V_n = 5x^2 \pm 1$ and $V_n = 7x^2 \pm 1$. In the present paper, when P is odd, we solve the equations $V_n = wx^2 \pm 1$ for $w = 10, 14, 15, 21, 35, 42, 70, 210$. We show that only V_1 can be of the form $wx^2 + 1$ and only V_1 or V_2 can be of the form $wx^2 - 1$.

We will use the Jacobi symbol throughout this study. Our method of proof is similar to that used by Cohn, Ribenboim and McDaniel in [3] and [6,7], respectively.

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2. Preliminaries

The following theorem is given in [8].

Theorem 2.1. *Let* $n \in \mathbb{N} \cup \{0\}$ *and* $m, r \in \mathbb{Z}$. *Then*

$$V_{2mn+r} \equiv (-1)^n V_r \pmod{V_m}. \tag{2.1}$$

If $n = 2 \cdot 2^k a + r$ with a odd, then we get

$$V_n = V_{2 \cdot 2^k a + r} \equiv -V_r (\bmod\, V_{2^k}) \tag{2.2}$$

by (2.1).

When P is odd, an induction method shows that

$$V_{2^k} \equiv 7 (\bmod\, 8)$$

and thus

$$\left(\frac{2}{V_{2^k}}\right) = 1 \tag{2.3}$$

and

$$\left(\frac{-1}{V_{2^k}}\right) = -1 \tag{2.4}$$

for all $k \geq 1$.

Moreover, if P is odd and $3 \nmid P$, then $V_{2^k} \equiv -1 (\bmod\, 3)$ and therefore

$$\left(\frac{3}{V_{2^k}}\right) = 1 \tag{2.5}$$

for all $k \geq 1$.

If P is odd and $3 | P$, then $V_{2^k} \equiv -1 (\bmod\, 3)$ and therefore

$$\left(\frac{3}{V_{2^k}}\right) = 1 \tag{2.6}$$

for all $k \geq 2$.

Thus (2.5) and (2.6) shows that

$$\left(\frac{3}{V_{2^k}}\right) = 1 \tag{2.7}$$

for all $k \geq 2$.

When P is odd, we have

$$\left(\frac{P-1}{V_{2^k}}\right) = \left(\frac{P+1}{V_{2^k}}\right) = 1 \tag{2.8}$$

for $k \geq 1$.

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