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On the genus of nil-graph of ideals of commutative rings

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Abstract. Let R be a commutative ring with identity and let $\mathrm{Nil}(R)$ be the ideal of all nilpotent elements of R. Let $\mathbb{I}(R) = \{I : I$ is a non-trivial ideal of R and there exists a non-trivial ideal J such that $IJ \subseteq \mathrm{Nil}(R)\}$. The *nil-graph* of ideals of R is defined as the simple undirected graph $\mathbb{AG}_N(R)$ whose vertex set is $\mathbb{I}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq \mathrm{Nil}(R)$. In this paper, we study the planarity and genus of $\mathbb{AG}_N(R)$. In particular, we have characterized all commutative Artin rings R for which the genus of $\mathbb{AG}_N(R)$ is either zero or one.

Keywords: Nil-graph of ideals; Commutative ring; Annihilating-ideal; Planar; Genus

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1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups, see [2,9,12]. The first graph construction from a commutative ring is the zero-divisor graph by Beck [9]. The zero-divisor graph was later studied by D.D. Anderson et al. [3] and Anderson et al. [2]. There are several other graphs associated with commutative rings such as the total graph [1], the annihilator graph [7] and the dot-product graph [8]. These consider the elements in the commutative ring as vertices. In ring theory, the structure of a ring R is more closely

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tied to its ideals behavior than to its elements, and so it is more appropriate to define a graph with ideals instead of elements as vertices. Some of the graph constructions with ideals of a commutative ring as vertices are the annihilating ideal graph [10] and the nil-ideal graph [16]. Several authors [4,5,19–23] studied various properties of these graphs including diameter, girth, domination and genus. In this paper, we are interested in certain topological properties of the nil-graph of ideals of commutative rings.

Throughout this paper, R is a commutative ring with identity which is not an integral domain. An ideal I of R is said to be an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = (0)$. We denote the set of non-zero annihilating ideals of R by $\mathbb{A}^*(R)$. Behboodi et al. [10,11] introduced and investigated the annihilating-ideal graph of R. The *annihilating-ideal graph* of R is defined as the simple undirected graph $\mathbb{AG}(R)$ whose vertex set is $\mathbb{A}^*(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Shaveisi et al. [16] generalized the annihilating-ideal graph of R and introduced the nil-graph of ideals of R. Let $\mathrm{Nil}(R)$ be the ideal of all nilpotent elements of R and $\mathbb{I}(R) = \{I : I \text{ is a non-trivial ideal of } R \text{ and there exists a non-trivial ideal } J \text{ such that } IJ \subseteq \mathrm{Nil}(R)\}$. The *nil-graph of ideals* of R is defined as the undirected simple graph $\mathbb{AG}_N(R)$ whose vertex set is $\mathbb{I}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq \mathrm{Nil}(R)$. It is easy to see that $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AG}_N(R)$.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The girth of G is the length of a shortest cycle in G and is denoted by $gr(G)$. If G has no cycles, we define the girth of G to be infinite. The *corona* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface with k handles. The genus of a graph G, denoted $g(G)$, is the smallest integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. We say that a graph G is planar if $g(G) = 0$, and toroidal if $g(G) = 1$. Note that a planar graph G has an embedding in the plane. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. Kuratowski's theorem says that a graph G is *planar* if and only if it contains no subdivision of K_5 or $K_{3,3}$. Also, if H is a subgraph of a graph G, then $g(H) \leq g(G)$. For details about the notion of embedding of a graph in a surface one can refer to A.T. White [24]. Several authors [6,13,17,18,23] studied the genus of graphs from commutative rings. In particular several characterizations are obtained for planar and toroidal nature of graphs from commutative rings. The purpose of this paper is to study the embeddings of the nil-graph of ideals $\mathbb{AG}_N(R)$. This paper is organized as follows.

In Section 2, we characterize all commutative Artin rings R for which the nil-graph of ideals $\mathbb{AG}_N(R)$ is planar. In Section 3, we characterize all commutative Artin rings R for which the nil-graph of ideals $\mathbb{AG}_N(R)$ is of genus one. Now we state a result which provides a characterization for $\mathbb{AG}_N(R)$ to be complete.