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Topics in differential geometry associated with position vector fields on Euclidean submanifolds

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Abstract. The position vector field is the most elementary and natural geometric object on a Euclidean submanifold. The purpose of this article is to survey six research topics in differential geometry in which the position vector field plays very important roles. In this article we also explain the relationship between position vector fields and mechanics, dynamics, and D'Arcy Thompson's law of natural growth in biology.

Keywords: Position vector field; Rectifying curve; Rectifying submanifold; Finite type submanifold; Ricci soliton; Biharmonic submanifold; Constant-ratio submanifold; Self-shrinker; Thompson's law of natural growth

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1. INTRODUCTION

For an n-dimensional submanifold M in the Euclidean m-space $\mathbb{E}^m$, the most elementary and natural geometric object is the position vector field $\mathbf{x}$ of M. The position vector, also known as *location vector* or *radius vector*, is a Euclidean vector $\mathbf{x} = \overrightarrow{OP}$ that represents the position of a point $P \in M$ in relation to an arbitrary reference origin O.

Among extrinsic invariants of a submanifold, the most natural and important one is the mean curvature vector H. In physics, the mean curvature vector field is the tension field imposed on the submanifold arising from the ambient space. In materials science, surface tension is used for either surface stress or surface free energy. It is well-known that surface tension is responsible for the shape of liquid droplets.

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The well-known formula of E. Beltrami (1835–1900) provides a simple relationship between the position vector field $\mathbf{x}$ and the mean curvature vector field H of M, namely,

$$\Delta \mathbf{x} = -nH, \tag{1.1}$$

where Δ denotes the Laplacian of M with respect to its induced metric on M (cf. [12,18,29, 33,89]).

A Euclidean submanifold is called a *minimal submanifold* if its mean curvature vector vanishes identically. The history of minimal submanifolds goes back to J.L. Lagrange (1736–1813) who initiated in 1760 the study of minimal surfaces in Euclidean 3-space (cf. [63]). Since then the theory of minimal surfaces has attracted many mathematicians for more than two centuries. In particular, minimal surfaces and minimal submanifolds in Riemannian manifolds of constant curvature have been investigated very extensively since then (see, e.g. [22,82,84]).

The position vector field also plays important roles in physics, in particular, in mechanics. In any equation of motion, the position vector $\mathbf{x}(t)$ is usually the most sought-after quantity because the position vector field defines the motion of a particle (i.e. a point mass)—its location relative to a given coordinate system at some time variable t. The first and the second derivatives of the position vector field with respect to time t give the velocity and acceleration of the particle.

The main purpose of this article is to survey six research topics in differential geometry in which the position vector field plays important roles. In this survey article we also explain the relationship between position vector fields and mechanics, dynamics, and D'Arcy Thompson's law of natural growth in biology.

2. Rectifying curves

In elementary differential geometry, most geometers describe a curve as a unit speed curve $\mathbf{x} = \mathbf{x}(s)$ whose position vector field is expressed in terms of an arc-length parameter s. In order to define curvature and torsion of a space curve, one needs the well-known Frenet formulas which can be obtained as follows:

Consider a unit-speed curve $\mathbf{x}: I \to \mathbb{E}^3$, defined on a real interval $I = (\alpha, \beta)$, that has at least four continuous derivatives. Put $\mathbf{t} = \mathbf{x}'(s)$. In general, it is possible that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can introduce a unique vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature* of the given curve $\mathbf{x}(t)$. Since $\mathbf{t}$ is a constant length vector field, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ which is a unit vector field orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ of the curve by the equation $\mathbf{b}' = -\tau\mathbf{n}$.

The famous *Frenet formulas* are given by

$$\begin{cases} \mathbf{t}' = \kappa\mathbf{n}, \\ \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b}' = -\tau\mathbf{n}. \end{cases} \tag{2.1}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$ and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively. A curve in $\mathbb{E}^3$ is called a *twisted curve* if it has nonzero curvature and nonzero torsion. A *helix* (or *curve of*