Arab Journal of Mathematical Sciences

Arab J Math Sci 23 (2017) 32–43

On connections on principal bundles

Indranil Biswas[1]

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

Received 12 July 2016; accepted 5 August 2016
Available online 24 August 2016

Abstract. A new construction of a universal connection was given in Biswas, Hurtubise and Stasheff (2012). The main aim here is to explain this construction. A theorem of Atiyah and Weil says that a holomorphic vector bundle E over a compact Riemann surface admits a holomorphic connection if and only if the degree of every direct summand of E is zero. In Azad and Biswas (2002), this criterion was generalized to principal bundles on compact Riemann surfaces. This criterion for principal bundles is also explained.

Keywords: Principal bundle; Universal connection; Holomorphic connection; Real Higgs bundle

2010 Mathematics Subject Classification: 53C05; 53C07; 32L05

1. Introduction

A connection ∇^0 on a C^∞ principal G-bundle $\mathcal{E}_G \longrightarrow \mathcal{X}$ is called *universal* if given any C^∞ principal G-bundle E_G on a finite dimensional C^∞ manifold M, and any connection ∇ on E_G, there is a C^∞ map

$$\xi : M \longrightarrow \mathcal{X}$$

such that

- the pulled back principal G-bundle $\xi^*\mathcal{E}_G$ is isomorphic to E_G, and
- the isomorphism between $\xi^*\mathcal{E}_G$ and E_G can be so chosen that it takes the pulled back connection $\xi^*\nabla^0$ on $\xi^*\mathcal{E}_G$ to the connection ∇ on E_G.

In [8] and [9] universal connections were constructed. In [4] a very simple, in fact quite tautological, universal connection was constructed.

E-mail address: indranil@math.tifr.res.in.

[1] The author acknowledges the support of a J. C. Bose Fellowship.
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2016.08.002

2. ATIYAH BUNDLE

All manifolds considered here will be C^∞, second countable and Hausdorff. Later we will impose further conditions such as complex structure.

Let G be a finite dimensional Lie group. Take a connected C^∞ manifold M. A principal G-bundle over M is a triple of the form

$$(E_G,\, p,\, \psi), \tag{2.1}$$

where

(1) E_G is a C^∞ manifold,

(2)

$$p\,:\, E_G \longrightarrow M \tag{2.2}$$

is a C^∞ surjective submersion, and

(3)

$$\psi\,:\, E_G \times G \longrightarrow E_G \tag{2.3}$$

is a C^∞ map defining a right action of G on E_G, such that the following two conditions hold:

- the two maps $p \circ \psi$ and $p \circ p_1$ from $E_G \times G$ to M coincide, where p_1 is the natural projection of $E_G \times G$ to E_G, and
- the map to the fiber product

$$\mathrm{Id}_{E_G} \times \psi\,:\, E_G \times G \longrightarrow E_G \times_M E_G$$

is a diffeomorphism; note that the first condition $p \circ \psi = p \circ p_1$ implies that the image of $\mathrm{Id}_{E_G} \times \psi$ is contained in the submanifold $E_G \times_M E_G \subset E_G \times E_G$ consisting of all points $(z_1,\, z_2) \in E_G \times E_G$ such that $p(z_1) = p(z_2)$.

Therefore, the first condition implies that G acts on E_G along the fibers of p, while the second condition implies that the action of G on each fiber of p is both free and transitive.

Take a C^∞ principal G-bundle $(E_G,\, p,\, \psi)$ over M. The tangent bundle of the manifold E_G will be denoted by TE_G. Take a point $x \in M$. Let

$$(TE_G)^x \,:=\, (TE_G)|_{p^{-1}(x)} \longrightarrow p^{-1}(x)$$

be the restriction of the vector bundle TE_G to the fiber $p^{-1}(x)$ of p over the point x. As noted above, the action ψ of G on E_G preserves $p^{-1}(x)$, and the resulting action of G on $p^{-1}(x)$ is free and transitive. Therefore, the action of G on TE_G given by ψ restricts to an action of G on $(TE_G)^x$. Let $\mathrm{At}(E_G)_x$ be the space of all G-invariant sections of $(TE_G)^x$. Since the action of G on the fiber $p^{-1}(x)$ is transitive, it follows that any G-invariant section of $(TE_G)^x$ is automatically smooth. More precisely, any G-invariant sections of $(TE_G)^x$ is uniquely determined by its evaluation of some fixed point of $p^{-1}(x)$. Therefore, $\mathrm{At}(E_G)_x$ is a real vector space whose dimension coincides with the dimension of E_G.

There is a natural vector bundle over M, which was introduced in [1], whose fiber over any $x \in M$ is $\mathrm{At}(E_G)_x$. This vector bundle is known as the *Atiyah bundle*, and it is denoted by $\mathrm{At}(E_G)$. We now recall the construction of $\mathrm{At}(E_G)$.