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Geometry of conformal vector fields

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Abstract. It is well known that the Euclidean space $(R^n, \langle,\rangle)$, the n-sphere $S^n(c)$ of constant curvature c and Euclidean complex space form $(C^n, J, \langle,\rangle)$ are examples of spaces admitting conformal vector fields and therefore conformal vector fields are used in obtaining characterizations of these spaces. In this article, we study the conformal vector fields on a Riemannian manifold and present the existing results as well as some new results on the characterization of these spaces. Taking clue from the analytic vector fields on a complex manifold, we define φ-analytic conformal vector fields on a Riemannian manifold and study their properties as well as obtain characterizations of the Euclidean space $(R^n, \langle,\rangle)$ and the n-sphere $S^n(c)$ of constant curvature c using these vector fields.

Keywords: Conformal vector fields; Ricci curvature; Scalar curvature; Obata's theorem; Laplacian; φ-analytic vector fields

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1. INTRODUCTION

Characterizations of important spaces such as the Euclidean space R^n, the Euclidean sphere S^n, and the complex projective space CP^n, is an important problem in differential geometry and was taken up by several authors (cf. [1–8,10–12,14,13,9,15–23], [26,24,27–32]). In most of these characterizations conformal vector fields play a notable role. Conformal vector fields are important objects for studying the geometry of several kinds of manifolds.

A smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if its flow consists of conformal transformations or, equivalently, if there exists a smooth function f on M (called the potential function of the conformal vector field ξ) that

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satisfies $\pounds_\xi g = 2fg$, where $\pounds_\xi g$ is the Lie derivative of g with respect ξ. We say that ξ is a nontrivial conformal vector field if ξ is a non-Killing vector field (ξ is a Killing vector field if the potential function $f = 0$ or, equivalently, the flow of ξ consists of isometries of the Riemannian manifold). If, in addition, ξ is a closed vector field (or is a gradient of a smooth function), then ξ is said to be a closed conformal vector field (or a gradient conformal vector field). If ξ is a gradient conformal vector field with $\xi = \nabla\rho$ for a smooth function ρ on the Riemannian manifold (M, g), then we get the Poisson equation $\varDelta\rho = nf$. Thus the geometry of gradient conformal vector fields on a Riemannian manifold is related to the Poisson equation on the Riemannian manifold. The role of differential equations in studying the geometry of a Riemannian manifold was initiated by the pioneering work of Obata (cf. [21–23]). The work of Obata is about characterizing specific Riemannian manifolds by second order differential equations. His main result is: a necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the n-sphere $S^n(c)$ is that there exists a non constant smooth function f on M that satisfies the differential equation $H_f = -cfg$, where H_f is the Hessian of the smooth function f. Then Tashiro [30] has shown that the Euclidean spaces R^n are characterized by the differential equation $H_f = cg$, and Tanno [28] obtained a similar characterization of spheres.

Recently Garcia-Rio et al. [15,16] have introduced the Laplace operator $\varDelta$ acting on smooth vector fields on a Riemannian manifold (M, g) and generalized the result of Obata using the differential equation satisfied by a vector field to characterize the n-sphere $S^n(c)$ (cf. Theorem 3.5 in [16]). These authors have also proved that the differential equation

$$\varDelta Z = -cZ, c = \frac{S}{n(n-1)},$$

where Z is a smooth vector field on an n-dimensional compact Einstein manifold (M, g) of constant scalar curvature $S > 0$, (that is Z is the eigenvector of the Laplace operator $\varDelta$), is a necessary and sufficient condition for M to be isometric to the n-sphere $S^n(c)$ (cf. Theorem 6 in [15]).

The geometry of conformal vector fields is naturally divided in two classes, the geometry of gradient conformal vector fields and the geometry of conformal vector fields which are not closed. The initial work on the subject of conformal vector fields was originated with the geometry of closed or gradient conformal vector fields. Riemannian manifolds admitting closed conformal vector fields or gradient conformal vector fields, have been investigated in [6,10,17–19,21–23,29–32] and it has been observed that there is a close relationship between the potential functions of gradient conformal vector fields and Obata's differential equation.

There are many examples of gradient conformal vector fields, on the n-dimensional sphere $S^n(c)$. If N is the unit normal vector field on $S^n(c)$, in the Euclidean space R^{n+1} with Euclidean metric $\langle , \rangle$, then for any constant vector field Z on the Euclidean space R^{n+1} its restriction to $S^n(c)$ can be expressed as $Z = \xi + fN$, where $f = \langle Z, N\rangle$ is a smooth function and ξ is a vector field on $S^n(c)$. Then it is straightforward to show that ξ is a gradient conformal vector field on $S^n(c)$ with potential function $-\sqrt{c}f$. Other classes of conformal vector gradient vector fields are provided by warped product spaces. For instance, consider an $(n-1)$-dimensional Riemannian manifold (M, g) and an open interval $I \subset R$ and set $\overline{M} = I \times M$ with projections $\pi_1 : \overline{M} \to I$ and $\pi_2 \to M$. Then for a positive function $f : I \to R$, we get the Riemannian manifold $(\overline{M}, \overline{g})$ called the warped product manifold,

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