ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Characterizing spheres by an immersion in Euclidean spaces[☆]

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Received 4 September 2016; accepted 9 September 2016

Abstract. In this paper we study compact immersed orientable hypersurfaces in the Euclidean space R^{n+1} and show that suitable restrictions on the tangential and normal components of the immersion give different characterizations of the spheres.

Keywords: Hypersurface; Conformal vector field; Ricci curvature; Position vector field

2010 Mathematics Subject Classification: 53C15; 53B25

1. INTRODUCTION

One of the interesting questions in the geometry of hypersurfaces in a Euclidean space is to find necessary and sufficient conditions for the hypersurface to be isometric to a sphere. Let M be an immersed orientable hypersurface of the Euclidean space R^{n+1} with the immersion $\psi : M \to R^{n+1}$. If N is the unit normal vector field to the hypersurface, then we can express ψ, (the position vector field of points of M in R^{n+1}), as $\psi = \xi + \rho N$, where ξ is vector field tangential to M and $\rho = \langle \psi, N \rangle$ is the support function of M, $\langle , \rangle$ being the Euclidean metric on R^{n+1}. The immersion ψ of the hypersurface M naturally gives two vector fields: ξ and $\bigtriangledown \rho$, the gradient of the support function ρ with respect to the induced metric g on the hypersurface M. One naturally expects that these vector fields play a vital role in shaping the geometry of a hypersurfaces. Since Killing vector fields and conformal vector fields have

☆ This work is supported by NSTIP strategic technologies program number (13-MAT1813-02) in the Kingdom of Saudi Arabia.

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Peer review under responsibility of King Saud University.

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http://dx.doi.org/10.1016/j.ajmsc.2016.09.002

been used in finding characterizations of spheres (cf. [1,2,5,3]), one would like to impose the conditions on these vector fields on the hypersurface to be Killing vector fields or conformal vector fields. However, the vector fields ξ and $\bigtriangledown\rho$ on the hypersurface M are both gradient vector fields ($\xi = \nabla f$, where $f = \frac{1}{2}\left\|\psi\right\|^2$) and if they are Killing vector fields they will be parallel and will not yield interesting results (cf. Remark 2.1). Therefore a natural choice is to consider these vector fields to be conformal vector fields. Conformal vector fields have been used in the study of hypersurfaces of a Riemannian manifold (cf. [4,1,6,7]). In particular it has been observed that the conformal vector field on the ambient space are closely related to the totally umbilical hypersurfaces. In this paper, we wish to put constraints on these vector fields and analyze the effects of these restrictions on the geometry of the hypersurface M. It is interesting to note that the restrictions on the vector fields ξ, $\bigtriangledown\rho$ on the compact hypersurface M, such as (i) ξ is a conformal vector field, (ii) $\bigtriangledown\rho$ is a conformal vector field, (iii) $\bigtriangledown\rho = \lambda\xi$, λ a constant (that is, $\bigtriangledown\rho$ is parallel to ξ), (iv) $\xi \perp \nabla\rho$, together with some suitable curvature restrictions, give respectively the characterizations of the spheres in R^{n+1} (cf. Theorems 3.1, 3.2, 4.1 and 4.2).

Trivial examples of such vector fields are the tangential and normal components of the natural embedding of the sphere $S^n(c)$ in the Euclidean space. As another example consider the warped product $M = (0,\infty) \times_t S^{n-1}$, where t is the coordinate function on $(0,\infty)$. Then the map $\varphi : M \to R^n - \{0\}$, $\varphi(t,x) = tx$ is an isometry and satisfies

$$d\varphi\left(\frac{\partial}{\partial t}\right) = \sum u^i \frac{\partial}{\partial u^i},$$

where $u^1,\ldots,u^n$ are the Euclidean coordinates on $R^n - \{0\}$. The isometric embedding $i : R^n - \{0\} \to R^{n+1}$, $i(x) = (x,0)$ gives the isometric embedding $\psi : M \to R^{n+1}$, $\psi = i \circ \varphi$ that satisfies

$$d\psi\left(t\frac{\partial}{\partial t}\right) = \sum tu^i \frac{\partial}{\partial u^i} = (\varphi(x),0)$$

and consequently the vector field ξ on the hypersurface M given by $\xi = t\frac{\partial}{\partial t}$ is easily seen to be a conformal vector field on M (cf. [8]).

2. Preliminaries

Let M be an orientable compact immersed hypersurface of the Euclidean space R^{n+1}. We denote by $\langle,\rangle$ the Euclidean metric on R^{n+1}, and by N, A and g the unit normal vector field, the shape operator and the induced metric on M respectively. If $\psi : M \to R^{n+1}$ is the immersion, then we have $\psi = \xi + \rho N$, with $\xi \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, and $\rho = \langle\psi, N\rangle$ is called the support function of the hypersurface M. We denote by $\bigtriangledown$ the covariant derivative operator with respect to the Riemannian connection on M. Taking the covariant derivative in equation $\psi = \xi + \rho N$ and using the Gauss Weingarten formulas of the hypersurface, we get

$$\nabla_X \xi = X + \rho AX \quad \text{and} \quad \nabla\rho = -A\xi \quad X \in \mathfrak{X}(M), \tag{2.1}$$

where $\bigtriangledown\rho$ is the gradient of ρ on the Riemannian manifold (M,g).