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Nonorientable manifolds, complex and symplectic structures, and characteristic classes

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A B S T R A C T

A complex manifold or a symplectic manifold is automatically oriented. We investigate these structures in the context of non-orientable manifolds. Any smooth connected non-orientable manifold is equipped with a real line bundle of order two. Various structures which are defined only on oriented manifolds extend to non-orientable manifolds once they are twisted by this line bundle of order two. Our aim is to develop this theme.

1. Introduction

An almost complex structure on a connected C^∞ manifold M is an automorphism J of the tangent bundle TM such that $J \circ J = -\mathrm{Id}_{TM}$. A necessary condition for the existence of an almost complex structure on M is that M is orientable. In fact, an almost complex structure produces an orientation of M. Now take M to be non-orientable. The

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two possible local orientations produce a real line bundle L on M equipped with a flat connection ∇ of order two, meaning $L^{\otimes 2}$ with the connection induced by ∇ is isomorphic to the trivial real line bundle on M equipped with the connection defined by the de Rham differential. We define an almost complex structure on M to be a C^∞ isomorphism of vector bundles

$$J : TM \longrightarrow TM \otimes L$$

such that the composition

$$TM \xrightarrow{J} TM \otimes L \stackrel{J\otimes \mathrm{Id}_L}{\longrightarrow} TM \otimes L \otimes L = TM$$

is $-\mathrm{Id}_{TM}$. There is a Nijenhuis tensor associated to an almost complex structure on M (see Section 5.1 for details). An almost complex structure is called integrable if the Nijenhuis tensor associated to it vanishes identically.

We show that the notions of a complex and a holomorphic vector bundle on a complex manifold extend to this context of integrable almost complex structure on a non-orientable manifold. A complex vector bundle on M is a pair (V, J_V), where V is a C^∞ real vector bundle on M and

$$J_V : V \longrightarrow V \otimes L$$

is a C^∞ isomorphism of vector bundles, such that the composition

$$V \xrightarrow{J_V} V \otimes L \stackrel{J_V\otimes \mathrm{Id}_L}{\longrightarrow} V \otimes L \otimes L = V$$

coincides with $-\mathrm{Id}_V$. A holomorphic vector bundle on M is a complex vector bundle on M equipped with an integrable Dolbeault operator (see Section 5.4 for details). An equivalent definition of a holomorphic vector bundle in terms of holomorphic transition functions is given in Section 5.3.

Let $\underline{L}$ denote that rank one real local system on M defined by the sheaf of flat sections of the above line bundle L equipped with flat connection ∇. For a complex vector bundle (V, J_V) on M, the i-th Chern class $c_i(V, J_V)$ is an element of the cohomology group $H^{2i}(M, \underline{L})$.

A symplectic structure on M is a “closed” and nondegenerate C^∞ two-form on M with values in the above line bundle L. A contact form on M can be defined similarly. We investigate these structures.

2. Orientation bundle and twisted differential forms

2.1. The orientation bundle

The multiplicative group $\mathbb{R} \setminus \{0\}$ will be denoted by $\mathbb{R}^*$. The connected component of $\mathbb{R}^*$ consisting of positive real numbers will be denoted by $\mathbb{R}^+$, so $\mathbb{R}^*/\mathbb{R}^+ = \mathbb{Z}/2\mathbb{Z}$. The