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Short-time asymptotics of the regularizing effect for semigroups generated by quadratic operators

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We study accretive quadratic operators with zero singular spaces. These degenerate non-selfadjoint differential operators are known to be hypoelliptic and to generate contraction semigroups which are smoothing in the Schwartz space for any positive time. In this work, we study the short-time asymptotics of the regularizing effect induced by these semigroups. We show that these short-time asymptotics of the regularizing effect depend on the directions of the phase space, and that this dependence can be nicely understood through the structure of the singular space. As a byproduct of these results, we derive sharp subelliptic estimates for accretive quadratic operators with zero singular spaces pointing out that the loss of derivatives with respect to the elliptic case also depends on the phase space directions according to the structure of the singular space. Some applications of these results are then given to the study of degenerate hypoelliptic Ornstein–Uhlenbeck operators and degenerate hypoelliptic Fokker–Planck operators.

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1. Introduction

1.1. Quadratic operators

We study in this work quadratic operators. This class of operators stands for pseudodifferential operators

$$q^w(x,D_x)u(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}q\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi, \tag{1.1}$$

defined by the Weyl quantization of complex-valued quadratic symbols

$$\begin{aligned} q:\mathbb{R}^{2n}&\to\mathbb{C}, \qquad n\geq 1,\\ (x,\xi)&\mapsto q(x,\xi). \end{aligned}$$

These non-selfadjoint operators are in fact only differential operators since the Weyl quantization of the quadratic symbol $x^\alpha\xi^\beta$, with $(\alpha,\beta)\in\mathbb{N}^{2n}$, $|\alpha+\beta|=2$, is simply given by

$$(x^\alpha\xi^\beta)^w=\mathrm{Op}^w(x^\alpha\xi^\beta)=\frac{x^\alpha D_x^\beta+D_x^\beta x^\alpha}{2}, \tag{1.2}$$

with $D_x=i^{-1}\partial_x$. The maximal closed realization of a quadratic operator $q^w(x,D_x)$ on $L^2(\mathbb{R}^n)$, that is, the operator equipped with the domain

$$D(q^w)=\big\{u\in L^2(\mathbb{R}^n):\ q^w(x,D_x)u\in L^2(\mathbb{R}^n)\big\}, \tag{1.3}$$

where $q^w(x,D_x)u$ is defined in the distribution sense, is known to coincide with the graph closure of its restriction to the Schwartz space [16] (pp. 425–426),

$$q^w(x,D_x):\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n).$$

Classically, to any quadratic form defined on the phase space

$$q:\mathbb{R}^n_x\times\mathbb{R}^n_\xi\to\mathbb{C},$$

is associated a matrix $F\in M_{2n}(\mathbb{C})$ called its Hamilton map, or its fundamental matrix, which is defined as the unique matrix satisfying the identity

$$\forall(x,\xi)\in\mathbb{R}^{2n},\forall(y,\eta)\in\mathbb{R}^{2n},\quad q\big((x,\xi),(y,\eta)\big)=\sigma\big((x,\xi),F(y,\eta)\big), \tag{1.4}$$

with $q(\cdot,\cdot)$ the polarized form associated to the quadratic form q, where σ stands for the standard symplectic form