Bull. Sci. math. 141 (2017) 312–352

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

A modulation technique for the blow-up profile of the vector-valued semilinear wave equation

Asma Azaiez[a,*,1,3], Hatem Zaag[b,2,4]

[a] *Université de Cergy-Pontoise, AGM, CNRS (UMR 8088), 95302, Cergy-Pontoise, France*
[b] *Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France*

A R T I C L E I N F O

Article history:
Received 20 March 2017
Available online 26 April 2017

Keywords:
Wave equation
Blow-up profile
Stationary solution
Modulation theory
Vector valued PDE

A B S T R A C T

We consider a vector-valued blow-up solution with values in $\mathbb{R}^m$ for the semilinear wave equation with power nonlinearity in one space dimension (this is a system of PDEs). We first characterize all the solutions of the associated stationary problem as an m-parameter family. Then, we show that the solution in self-similar variables approaches some particular stationary one in the energy norm, in the non-characteristic cases. Our analysis is not just a simple adaptation of the already handled real or complex case. In particular, there is a new structure of the set of stationary solutions.

* Corresponding author.
E-mail addresses: asma.azaiez@u-cergy.fr (A. Azaiez), zaag@math.univ-paris13.fr (H. Zaag).

[1] Supported by the ERC Advanced Grant no. 291214, BLOWDISOL.

[2] Supported by the ERC Advanced Grant no. 291214, BLOWDISOL and by ANR project ANAÉ ref. ANR-13-BS01-0010-03.

[3] Address: Université de Cergy-Pontoise, Laboratoire Analyse Géometrie Modélisation, CNRS-UMR 8088, 2 avenue Adolphe Chauvin 95302, Cergy-Pontoise, France.

[4] Address: Université Paris 13, Institut Galilée, Laboratoire Analyse Géometrie et Applications, CNRS-UMR 7539, 99 avenue J.B. Clément 93430, Villetaneuse, France.

http://dx.doi.org/10.1016/j.bulsci.2017.04.001
0007-4497/

1. Introduction

We consider the vector-valued semilinear wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where here and all over the paper $|.|$ is the euclidian norm in $\mathbb{R}^m$, $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}^m$, $m \geq 2$, $p > 1$, $u_0 \in H^1_{loc,u}$ and $u_1 \in L^2_{loc,u}$ with $||v||^2_{L^2_{loc,u}} = \sup\limits_{a\in\mathbb{R}} \int\limits_{|x-a|<1} |v(x)|^2 dx$ and $||v||^2_{H^1_{loc,u}} = ||v||^2_{L^2_{loc,u}} + ||\nabla v||^2_{L^2_{loc,u}}$.

The Cauchy problem for equation (1) in the space $H^1_{loc,u} \times L^2_{loc,u}$ follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$. See for instance Ginibre, Soffer and Velo [9], Ginibre and Velo [10], Lindblad and Sogge [14] (for the local in time wellposedness in $H^1 \times L^2$). Existence of blow-up solutions follows from ODE techniques or the energy-based blow-up criterion of [13]. More blow-up results can be found in Caffarelli and Friedman [6], Alinhac [1] and [2], Kichenassamy and Littman [12,11], Shatah and Struwe [25].

The real case (in one space dimension) has been understood completely, in a series of papers by Merle and Zaag [18,19,21,22] and in Côte and Zaag [7] (see also the note [20]). Recently, the authors give an extension to higher dimensions in [24] and [23], where the blow-up behavior is given, together with some stability results.

For other types of nonlinearities, we mention the recent contribution of Azaiez, Masmoudi and Zaag in [5], where we study the semilinear wave equation with exponential nonlinearity, in particular we give the blow-up rate with some estimations.

In [4], we consider the complex-valued solution of (1) (or $\mathbb{R}^2$-valued solution), characterize all stationary solutions and give a trapping result. The main obstruction in extending those results to the vector case $m \geq 3$ was the question of classification of all self similar solutions of (1) in the energy space. In this paper we solve that problem and show that the real valued and complex valued classification also hold in the vector-valued case $m \geq 3$ (see Proposition 2 below), with an adequate choice in S^{m-1}. This is in fact our main contribution in this paper, and it allows us to generalize the results of the complex case to the vector valued case $m \geq 3$. In this paper, we aim at proving similar results for the general case $u(x,t) \in \mathbb{R}^m$, for $m \geq 3$.

Let us first introduce some notations before stating our results.

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous curve Γ as the graph of a function $x \to T(x)$ such that the domain of definition of u (or the maximal influence domain of u) is

$$D_u = \{(x,t) | t < T(x)\}.$$