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Limit theorems for sub-sums of partial quotients of continued fractions

Liangang Ma[a], Radhakrishnan Nair[b,*]

[a] *Department of Mathematical Sciences, Binzhou University, Huanghe 5 road No. 391, City of Binzhou, Shandong Province, PR China*
[b] *Department of Mathematical Sciences, The University of Liverpool, Peach Street, Liverpool L69 7ZL, United Kingdom*

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Abstract

This paper studies the limit behaviour of sums of the form

$$T_n(x) = \sum_{1 \le j \le n} c_{k_j}(x), (n = 1, 2, \ldots)$$

where $(c_j(x))_{j \ge 1}$ is the sequence of partial quotients in the regular continued fraction expansion of the real number x and $(k_j)_{j \ge 1}$ is a strictly increasing sequence of natural numbers. Of particular interest is the case where for irrational α, the sequence $(k_j \alpha)_{j \ge 1}$ is uniformly distributed modulo one and $(k_j)_{j \ge 1}$ is good universal. It was observed by the second author, for this class of sequences $(k_j)_{j \ge 1}$ that we have $\lim_{n \to \infty} \frac{T_n(x)}{n} = +\infty$ almost everywhere with respect to Lebesgue measure. The case $k_j = j$ $(j = 1, 2, \ldots)$ is classical and due to A. Ya. Khinchin. Building on work of H. Diamond, Khinchin, W. Philipp, L. Heinrich, J. Vaaler and others, in the special case where $k_j = j$ $(j = 1, 2, \ldots,)$ we examine the asymptotic behaviour of the sequence $(T_n(x))_{n \ge 1}$ in more detail.

Keywords: Partial quotients; Distribution functions; Mixing

* Corresponding author.
E-mail addresses: maliangang000@163.com (L. Ma), nair@liverpool.ac.uk (R. Nair).

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1. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of natural numbers. For $x \in (0, 1)$, let $x = [c_1(x), c_2(x), \ldots]$ denote its regular continued fraction expansion. Recall that we say a sequence $(x_n)_{n\geq 1}$ is uniformly distributed modulo one if for each interval $I \subseteq [0, 1)$ of length $|I|$ we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : x_n \in I\} = |I|.$$

Here for a finite set F we have used $\#F$ to denote its cardinality. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measurable map, that is also measure-preserving. That is, given $A \in \mathcal{B}$, we have $\mu(T^{-1}A) = \mu(A)$, where $T^{-1}A$ denotes the set $\{x \in X : Tx \in A\}$. We call $(X, \mathcal{B}, \mu, T)$ a dynamical system. We say a dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic if $T^{-1}A = A$ for $A \in \mathcal{B}$ means that either $\mu(A)$ or $\mu(X \setminus A)$ is 0. We say $(k_n)_{n\geq 0}$ is L^p good universal if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and for each $f \in L^p(X, \mathcal{B}, \mu)$ the limit

$$\ell_{T,f}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n}x),$$

exists μ almost everywhere.

For a real number y let $[y]$ denote the largest integer not greater than y. Also let $\{y\}$ denote the fractional part of y i.e. $y - [y]$. We call

$$G(x) = \begin{cases} \left\{\frac{1}{x}\right\}, & \text{if } x \in (0, 1) \\ 0 & \text{if } x = 0 \end{cases}$$

the Gauss map. Let ρ_L denote Lebesgue measure on $[0, 1)$. Set

$$\rho_G(A) = \frac{1}{\log 2} \int_A \frac{dx}{1 + x}$$

for a ρ_L-measurable set A. We call ρ_G the Gauss measure.

Let $\mathcal{M}$ denote the Lebesgue σ- algebra on $[0, 1)$. Applying good universality to the dynamical system $([0, 1), \mathcal{M}, \rho_G, G)$, using the fact that

$$c_1(x) = \left[\frac{1}{x}\right], \quad c_{k+1}(x) = c_k(G(x)), \quad (k = 1, 2, \ldots)$$

for irrational x in [14], developing ideas in [6] and [19], the following is proved.

Suppose that the function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and increasing and that for some $p \geq 1$ we have

$$\int_0^1 \frac{|F(c_1(x))|^p}{x + 1} dx < \infty.$$

Suppose (i) for each irrational α that $(\{k_j\alpha\})_{j\geq 1}$ is uniformly distributed modulo one, and (ii) that $(k_j)_{\geq 1}$ is L^p good universal. For a finite set of non-negative real numbers $\{a_1, \ldots, a_n\}$ we let

$$M_{F,n}(a_1, \ldots, a_n) = F^{-1}\left[\frac{F(a_1) + \cdots + F(a_n)}{n}\right].$$

It is shown in [14] that

$$\lim_{n\to\infty} M_{F,n}(c_1(x), \ldots, c_n(x)) = F^{-1}\left[\frac{1}{\log 2} \int_0^1 \frac{F(c_1(x))}{x + 1} dx\right]$$