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# An introduction of logical entropy on sequential effect algebra

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**Abstract**

The main aim of this study was to introduce logical entropy on dynamical systems that their state spaces were sequential effect algebra. In this regard, logical partition was defined on sequential effect algebra and then based on logical partition concept, logical entropy on partitions, conditional logical entropy, and logical entropy on dynamical systems were introduced and their features were analyzed. In addition, it was proved that this entropy is an invariant object under isomorphism relation.


*Keywords:* Logical entropy; Sequential effect algebra; Dynamical system

## 1. Introduction and preliminaries

Since the introduction of entropy concept by Clasius in 1854 and until now, various entropies have been introduced for different purposes and have been used in different scientific fields. Kolmogorov offered the primary concepts for definition of entropy on probability space [7]. Then Sinai expanded this concept and introduced entropy on dynamical system that its state space is probability space [9]. This entropy was used as a tool for measuring the uncertainty quantity for random variable. Topological entropy was also another entropy which was introduced in mathematics field [1]. This entropy measures the complex behavior of the orbits in a dynamical

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system during the time. Shannon in a paper "naming mathematical communication theory" offered the entropy concept in Information Theory [8]. This entropy is a criterion for random mistakes which are formed while transferring a signal. Recently logical entropy has been formed by Ellerman based on logical partitioning [4]. The logical concept of entropy based on partition logic is the normalized counting measure of the set of distinctions of a partition on a finite set just as the usual logical notion of probability based on the Boolean logic of subsets is the normalized counting measure of the subsets (events) [5]. Many researchers had defined metric entropy on dynamical systems that their state space is an algebra structure [2,3,6]. One of these structures is effect algebras. The effect algebras are algebraic structure which is used in quantum physics and the definition of entropy on this structure is really important. In this study, logical entropy on dynamical systems that their state spaces were sequential effect algebra was introduced and its properties were discussed.

## 2. Logical entropy of partitions on sequential effect algebra

In this section, at first concepts of logical partition, the join and interior subset of two logical partitions and the refinement of a logical partition of a sequential effect algebra were defined. Then the notion of logical entropy on a logical partition was introduced and the relations between entropies of a logical partition, the refinement of a logical partition, the join and interior subset of two logical partitions were investigated.

**Definition 2.1.** Let $(E, \oplus, \circ, \theta, 1)$ be an algebra, $\oplus$, $\circ$ be partial operations on $E$ and $\theta, 1$ be constant elements in $E$. This algebra is called sequential effect algebra (SEA in short) if for all $a, b, c \in E$, the following conditions hold:

(I) if $a \oplus b$ and $(a \oplus b) \oplus c$ are defined then $b \oplus a$, $a \oplus (b \oplus c)$, $c \circ a \oplus c \circ b$ and $a \circ c \oplus b \circ c$ are defined and $a \oplus b = b \oplus a$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, $c \circ (a \oplus b) = c \circ a \oplus c \circ b$ and $(a \oplus b) \circ c = a \circ c \oplus b \circ c$;
(II) for any $a \in E$, there exists a unique element $a' \in E$ such that $a \oplus a' = 1$ and $1 \circ a = a$;
(III) if $a \oplus 1$ is defined in E, then $a = \theta$;
(IV) if $a \circ b = \theta$, then $a \circ b = b \circ a$;
(V) if $a \circ b = b \circ a$, then $a \circ b\prime = b\prime \circ a$ and for each $c \in E$, $a \circ (b \circ c) = (a \circ b) \circ c$;
(VI) if $c \circ a = a \circ c$ and $c \circ b = b \circ c$, then $c \circ (a \circ b) = (a \circ b) \circ c$ and $c \circ (a \oplus b) = (a \oplus b) \circ c$ whenever $a \oplus b$ is defined;
$a \leq c$ if and only if, there exists an element $b \in E$ such that $a \oplus b = c$.

**Definition 2.2.** A subset $P = \{p_i : i = 1, \ldots, n\}$ of a SEA $(E, \oplus, \circ, \theta, 1)$ is called a partition if $\sum_{i=1}^{n} p_i$ exists in $E$ and $\sum_{i=1}^{n} p_i = 1$. Partition $Q = \{q_j : j = 1, \ldots, m\}$ is a refinement of the partition $P = \{p_i : i = 1, \ldots, n\}$, if for every $p_i$ there is a subset $\eta_i \subseteq \{1, \ldots, m\}$ such that $p_i = \sum_{j \in \eta_i} q_j$ and $\cup_{i=1}^{n} \eta_i = \{1, \ldots, m\}$, $\eta_i \cap \eta_j = \emptyset$ for every $i \neq j$ and we write $P \prec Q$.

**Definition 2.3.** Let $E$ be a SEA. A mapping $m : E \longrightarrow [0, 1]$ is called a state if

(I) m(1)=1;
(II) if $a \oplus b$ exists and $a \oplus b = e$, then $m(a \oplus b) = m(e) = m(a) + m(b)$;
(III) $m(a \circ b) \geq m(a)m(b)$.

**Proposition 2.4.** *Let $(E, \oplus, \circ, \theta, 1)$ be a SEA, $P = \{p_i\}_{i=1}^{n}$ and $Q = \{q_j\}_{j=1}^{m}$ be two partitions of $E$, then:*