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An elementary representation of the higher-order Jacobi-type differential equation

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Abstract

We investigate the differential equation for the Jacobi-type polynomials which are orthogonal on the interval $[-1, 1]$ with respect to the classical Jacobi measure and an additional point mass at one endpoint. This scale of higher-order equations was introduced by J. and R. Koekoek in 1999 essentially by using special function methods. In this paper, a completely elementary representation of the Jacobi-type differential operator of any even order is given. This enables us to trace the orthogonality relation of the Jacobi-type polynomials back to their differential equation. Moreover, we establish a new factorization of the Jacobi-type operator into a product of linear second-order operators, which gives rise to a recurrence relation with respect to the order of the equation. Finally, two interrelations with the differential equation for the symmetric ultraspherical-type polynomials are pointed out.

Keywords: orthogonal polynomials; Higher-order linear differential equations; Jacobi-type equations; Jacobi-type polynomials; Factorization

1. Introduction and main result

In 1999, J. and R. Koekoek [6] established a new class of higher-order linear differential equations satisfied by the generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}, \alpha, \beta > -1, M, N \geq 0$. These function systems were introduced and studied by T. H. Koornwinder [8] as the orthogonal polynomials with respect to a linear combination of the Jacobi weight function

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$w_{\alpha,\beta}$ and one or two delta functions at the endpoints of the interval $-1 \le x \le 1$,

$$\begin{aligned} w_{\alpha,\beta,M,N}(x) &= w_{\alpha,\beta}(x) + M\delta(x+1) + N\delta(x-1),\\ w_{\alpha,\beta}(x) &= h^{-1}_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta,\\ h_{\alpha,\beta} &= \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta dx = 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)/\Gamma(\alpha+\beta+2). \end{aligned} \tag{1.1}$$

In the present paper we investigate the so-called Jacobi-type equation with one additional mass point in the weight function, i.e. with either M or N being positive. Recently, we extended the main result to the most general case that M and N are both nonzero, see [14]. In terms of the classical Jacobi polynomials [3, Sec. 10.8]

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \tag{1.2}$$

for $n \in \mathbb{N}_0 = \{0, 1, \ldots\}$, the Jacobi-type polynomials are given by, cf. [6, (56), (58)] together with [3, 10.8(17)],

$$P_n^{\alpha,\beta,M,N}(x) = P_n^{\alpha,\beta}(x) + MQ_n^{\alpha,\beta}(x) + NR_n^{\alpha,\beta}(x),\ n\in\mathbb{N}_0,\ M\cdot N = 0, \tag{1.3}$$

where, with $A_n^{\alpha,\beta} = (\alpha+2)_{n-1}(\alpha+\beta+2)_n/[2n!(\beta+1)_{n-1}]$,

$$Q_n^{\alpha,\beta}(x) = A_n^{\beta,\alpha}(x+1)P_{n-1}^{\alpha,\beta+2}(x),\ n\in\mathbb{N},\ Q_0^{\alpha,\beta}(x) = 0, \tag{1.4}$$

$$R_n^{\alpha,\beta}(x) = A_n^{\alpha,\beta}(x-1)P_{n-1}^{\alpha+2,\beta}(x),\ n\in\mathbb{N},\ R_0^{\alpha,\beta}(x) = 0. \tag{1.5}$$

By means of the well-known relationship $P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x)$, it follows that, [8, (2.5)], [6, (52)],

$$Q_n^{\alpha,\beta}(x) = (-1)^n R_n^{\beta,\alpha}(-x),\ P_n^{\alpha,\beta,M,0}(x) = (-1)^n P_n^{\beta,\alpha,0,M}(-x),\ n\in\mathbb{N}_0, M>0. \tag{1.6}$$

Hence it suffices to treat, for instance, the case $M = 0$, $N > 0$ in full detail. The corresponding results for $M > 0$, $N = 0$ then follow immediately.

For $\alpha \in \mathbb{N}_0$ and any $\beta > -1$, $N > 0$, J. and R. Koekoek [6, Sec. 2] found that the Jacobi-type polynomials $\{P_n^{\alpha,\beta,0,N}(x)\}_{n=0}^\infty$ satisfy a linear differential equation of order $2\alpha+4$ which, for our purpose, is conveniently described in the form

$$N\left\{L_{2\alpha+4,x}^{\alpha,\beta} - \Lambda_{2\alpha+4,n}^{\alpha,\beta}\right\}y(x) + C_{\alpha,\beta}\left\{L_{2,x}^{\alpha,\beta} - \Lambda_{2,n}^{\alpha,\beta}\right\}y(x) = 0,\ -1<x<1. \tag{1.7}$$

Here, the eigenvalue parameters and the coupling constant read

$$\begin{aligned} \Lambda_{2\alpha+4,n}^{\alpha,\beta} &= (n)_{\alpha+2}(n+\beta)_{\alpha+2},\ \Lambda_{2,n}^{\alpha,\beta} = n(n+\alpha+\beta+1),\\ C_{\alpha,\beta} &= (\alpha+2)!(\beta+1)_{\alpha+1}\,. \end{aligned} \tag{1.8}$$

In particular, when N tends to zero, Eq. (1.7) reduces to the classical equation for the Jacobi polynomials, $L_{2,x}^{\alpha,\beta}P_n^{\alpha,\beta}(x) = \Lambda_{2,n}^{\alpha,\beta}P_n^{\alpha,\beta}(x)$, $n\in\mathbb{N}_0$, where

$$\begin{aligned} L_{2,x}^{\alpha,\beta}y(x) &= \{(x^2-1)D_x^2 + [\alpha-\beta+(\alpha+\beta+2)x]\,D_x\}y(x)\\ &= (x-1)^{-\alpha}(x+1)^{-\beta}D_x[(x-1)^{\alpha+1}(x+1)^{\beta+1}D_x y(x)]. \end{aligned} \tag{1.9}$$

Throughout this paper, $D_x^i \equiv (D_x)^i$ denotes the i-fold differentiation with respect to x. The crucial part of Eq. (1.7) was to determine the coefficient functions in the higher-order differential expression

$$L_{2\alpha+4,x}^{\alpha,\beta}y(x) = \sum_{i=1}^{2\alpha+4} d_i^{\alpha,\beta}(x)D_x^i y(x) \tag{1.10}$$

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