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Burkholder Inequalities in Riesz spaces

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Abstract

In this paper, the extent to which the Burkholder Inequalities in classical Stochastic Analysis can be generalized to the new Theory of Stochastic Analysis in Riesz spaces.

Keywords: Riesz space; Conditional expectation on Riesz spaces; Martingale; Burkholder Inequality; Stochastic process in Riesz spaces

1. Introduction

Almost a decade ago, the Theory of Stochastic Analysis on Riesz Spaces has been introduced by Kuo, Labuschagne, and Watson [13,14]. Since then, much attention has been paid to this theory mainly by the South African School itself (see, e.g., [2,3,8–12,17,18]). This paper aims to give a contribution to this program. More precisely, we are interested in one of the major inequalities in Martingale Theory, *viz.*, the classical Burkholder's inequality [4,6]. We recall some of the relevant ideas.

Let $\{(X_n, \mathcal{F}_n) : n \geq 1\}$ be a martingale. Martingale increments are given by

$$\Delta X_1 = X_1 \quad \text{and} \quad \Delta X_n = X_n - X_{n-1} \quad \text{for all } n = 2, 3, \ldots$$

and the Quadratic Variation Process is defined by

$$S_n(X) = (\Delta X_1)^2 + \cdots + (\Delta X_n)^2 \quad \text{for all } n = 1, 2, \ldots.$$

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Roughly speaking, the Burkholder inequality stipulates that, as far as L^p-norms are concerned, $S_n^{1/2}$ and X_n increase at the same rate. More precisely, for every $p \in (1, \infty)$ there do exist positive real numbers a_p and b_p such that

$$a_p \left\| S_n^{1/2} \right\|_p \leq \|X_n\|_p \leq b_p \left\| S_n^{1/2} \right\|_p .$$

Recently, Troitsky asked us (in a private communication) to investigate the extent to which this inequality can be generalized to the more general setting of Stochastic Analysis in Riesz spaces (i.e., vector lattices). More details about this question seem in order.

Throughout this paper, T stands for a conditional expectation with natural domain $L^1(T)$. Notice here that $L^1(T)$ is a Dedekind complete Riesz space with a weak order unit $e > 0$ and $Te = e$ (see [13] for conditional expectations on Riesz spaces and definition of natural domain). For $p \in (1, \infty)$ and $x \in L^p(T)^+$, we consider the p-power x^p as recently defined by Grobler in [9]. It should be pointed out that x^p lies in the universal completion of $L^1(T)$ (see [1] for the universally completion of an Archimedean Riesz space). Following [3], we put

$$L^p(T) = \left\{ x \in L^1(T) : |x|^p \in L^1(T) \right\}$$

and

$$N_p(x) = T\left(|x|^p\right)^{1/p} \quad \text{for all } x \in L^p(T).$$

As a matter of fact, $L^p(T)$ is a Riesz subspace of $L^1(T)$. Now, choose $p \in (1, \infty)$ and consider a family $\{T_n : n \geq 1\}$ of conditional expectations with $T_1 = T$ and $T_i T_j = T_j T_i = T_i$ whenever $i \leq j$. Such a family is called a *filtration* in [14, Definition 3.1]. Furthermore, a *martingale* is defined in [14, Definition 3.2] to be a family $\{(x_n, T_n) : n \geq 1\}$ where $\{T_n : n \geq 1\}$ is a filtration and $x_n \in R(T_n)$ with

$$T_i\left(x_j\right) = x_i \quad \text{for all } i, j \quad \text{with } i \leq j.$$

Here, $R(T_n)$ denotes the range of T_n. Keeping the same notations as previously used in the concrete case, it turns out that there exist positive real numbers a_p and b_p such that

$$a_p N_p\left(S_n^{1/2}\right) \leq N_p(x_n) \leq b_p N_p\left(S_n^{1/2}\right).$$

The proof of this inequality is very technical in nature. Indeed, it is based upon a generalization of the standard Stopping Time and an integral representation of p-powers. We shall take the opportunity to review some calculation step in the paper [9, Page 18] which is quite unclear and somehow mysterious.

Finally, we shall use the books [6,16] as unique sources of unexplained terminology and notation on Probability Theory and Riesz Spaces, respectively.

2. Preliminaries

In order to avoid unnecessary repetition, beginning with the next lines and throughout the paper, E is assumed to be a Dedekind complete Riesz space with weak order unit $e > 0$.

A *filtration* in E is a sequence $(T_i)_{i \geq 1}$ of conditional expectations on E such that $T_i T_j = T_j T_i = T_j$ whenever $j \leq i$. Before giving the definition of stopping time let us recall some basic properties of band projections. A band B in a Riesz space E that satisfies $E = B \oplus B^d$ is referred to as a *projection band*. Clearly, each projection band B gives rise to a natural projection P with range B which is referred to as a band projection and denoted by P_B. If B_x is the principal

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