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Diffeomorphism groups of compact convex sets

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Abstract

For $K \subseteq \mathbb{R}^n$ a compact convex subset with non-empty interior, let $\mathrm{Diff}_{\partial K}(K)$ be the group of all C^∞-diffeomorphisms of K which fix ∂K pointwise. We show that $\mathrm{Diff}_{\partial K}(K)$ is a C^0-regular infinite-dimensional Lie group. As a byproduct, we obtain results concerning solutions to ordinary differential equations on compact convex sets.

Keywords: Compact convex set; Diffeomorphism group; Lie group; Flow; Regularity; Time-dependent vector field; Initial value problem; Existence; Non-open set; Picard iteration; Dependence on parameters; Inverse function

0. Introduction and statement of the main results

Lie groups of smooth diffeomorphisms of compact manifolds (like the diffeomorphism group $\mathrm{Diff}(\mathbb{S}^1)$ of the circle) are among the most prominent and important examples of infinite-dimensional Lie groups (see, e.g., [15,17,19,21,25]; cf. [26]). A Lie group structure on $\mathrm{Diff}(K)$ even is available if K is a compact manifold with boundary or corners [20]; this includes the case that $K \subseteq \mathbb{R}^2$ is a convex polyhedron.[1] In this article, we describe Lie groups of diffeomorphisms of an arbitrary compact convex subset $K \subseteq \mathbb{R}^n$ with non-empty interior (whose boundary ∂K

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[1] Diffeomorphism groups of non-compact manifolds can also be treated; their Lie group structures are modelled only on Lie algebras of compactly supported smooth vector fields. For Lie groups of real analytic diffeomorphisms of real analytic, compact manifolds with or without boundary or corners, cf. [6,7], and [17].

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need not satisfy any regularity assumptions). To explain the result, let us call a map $\gamma : K \to \mathbb{R}^n$ *smooth* if it is continuous, its restriction $\gamma|_{K^0}$ to the interior of K is smooth, and all iterated directional derivatives on K^0 admit continuous extensions to all of K (see 1.3 for details). We write $\mathrm{Diff}(K)$ for the group of all smooth diffeomorphisms of K, i.e., bijections $\phi : K \to K$ such that both ϕ and ϕ^{-1} are smooth in the preceding sense. We endow the space $C^\infty(K, \mathbb{R}^n)$ of all smooth $\mathbb{R}^n$-valued mappings on K with the smooth compact-open topology (as recalled in 1.4), which makes it a Fréchet space (see [14], cf. [2]). Then also the closed vector subspace

$$C^\infty_{\partial K}(K, \mathbb{R}^n) := \{\eta \in C^\infty(K, \mathbb{R}^n) : \eta|_{\partial K} = 0\}$$

of $C^\infty(K, \mathbb{R}^n)$ is a Fréchet space. Now

$$\mathrm{Diff}_{\partial K}(K) := \{\phi \in \mathrm{Diff}(K) : (\forall x \in \partial K) : \phi(x) = x\}$$

is a subgroup of $\mathrm{Diff}(K)$. We show that

$$\Omega := \{\phi - \mathrm{id}_K : \phi \in \mathrm{Diff}_{\partial K}(K)\}$$

is an open 0-neighbourhood in $C^\infty_{\partial K}(K, \mathbb{R}^n)$ (see Section 3), enabling us to consider $\mathrm{Diff}_{\partial K}(K)$ as a smooth manifold modelled on $C^\infty_{\partial K}(K, \mathbb{R}^n)$ with

$$\mathrm{Diff}_{\partial K}(K) \to \Omega, \quad \phi \mapsto \phi - \mathrm{id}_K$$

as a global chart. As our main result, we obtain (see Sections 4–6):

Theorem A. $\mathrm{Diff}_{\partial K}(K)$ *is a C^0-regular Lie group.*

Recall that, if G is a Lie group modelled on a locally convex space E, with multiplication $\mu : G \times G \to G$, then the tangent map $T\mu : T(G \times G) \cong TG \times TG \to TG$ restricts to a smooth right action

$$TG \times G \to TG, \quad (v, g) \mapsto v.g$$

(identifying G with the zero-section in TG). Let $\mathfrak{g} := L(G) := T_{\mathbf{e}}G \cong E$ be the Lie algebra of G (the tangent space at the neutral element $\mathbf{e}$). The Lie group G is called C^0-*regular* if for each $\gamma \in C([0, 1], \mathfrak{g})$, there is a (necessarily unique) C^1-curve $\mathrm{Evol}^r(\gamma) := \eta : [0, 1] \to G$ such that $\eta(0) = \mathbf{e}$ and

$$\eta'(t) = \gamma(t).\eta(t) \quad \text{for all } t \in [0, 1],$$

and moreover the map

$$\mathrm{evol}^r : C([0, 1], \mathfrak{g}) \to G, \quad \gamma \mapsto \mathrm{Evol}^r(\gamma)(1)$$

is smooth (using the compact-open topology on the left); cf. [5,8,22].

If G is C^0-regular, then G is *regular* (i.e., it has the analogous property with $C^\infty([0, 1], \mathfrak{g})$ in place of $C([0, 1], \mathfrak{g})$). Regularity is a central concept in infinite-dimensional Lie theory, and needed as a hypothesis in many results of this theory. We refer to [21] and [22] for more information (cf. also [17]). Proofs for regularity properties of diffeomorphism groups can be found, e.g., in [15,17,18,21,27], and [28].

Previously, mappings of the form $\phi \mapsto \phi - \mathrm{id}$ have been used as a global chart for the Lie group $\mathrm{Diff}_c(\mathbb{R}^n)$ of compactly supported smooth diffeomorphisms of $\mathbb{R}^n$ [12], for certain weighted diffeomorphism groups of $\mathbb{R}^n$ (like Lie groups of rapidly decreasing diffeomorphisms) [28], and for further more specialized diffeomorphism groups [18].

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