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On the quotient class of non-archimedean fields

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Abstract

The *quotient class* of a non-archimedean field is the set of cosets with respect to all of its additive convex subgroups. The algebraic operations on the quotient class are the Minkowski sum and product. We study the algebraic laws of these operations. Addition and multiplication have a common structure in terms of regular ordered semigroups. The two algebraic operations are related by an adapted distributivity law.

Keywords: Non-archimedean fields; Cosets; Regular semigroups; Convexity

1. Introduction

We study the algebraic properties of the set of cosets with respect to all possible convex additive subgroups of a non-archimedean field F, typically a field of formal series or a Hardy-field. We will call this set of cosets the *quotient class* of F. Because the (Minkowski) sum of a nontrivial convex additive subgroup and an arbitrary element can never be zero, the quotient class cannot be a group for addition, and for similar reasons neither for multiplication. Still a quotient class satisfies rather strong algebraic properties, for, as we will see, addition and multiplication are commutative, satisfy the properties of regular semigroups and are related by an adapted distributive law.

The common structure of addition and multiplication is stronger than a regular semigroup and was called *assembly* in [7]. We will call *magnitude* a convex additive subgroup M of F,

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this is in line with a common interpretation of Hardy-fields as models of orders of magnitude of functions [1,3,8]. There exists a definite relationship between non-archimedean structures and asymptotics [2,4,12,15]; in a sense, a magnitude may be seen as the size of an imprecision. Given a coset with respect to M, the magnitude M acts as an individualized neutral element for addition. If α is a coset which is not reduced to a magnitude M it has an individualized neutral element for multiplication $1 + M/\alpha$, which with some abuse of language is called *unity*.

It is easy to see that distributivity does not hold in general. However we show that distributivity holds up to a correction term which has the form of a magnitude. We will identify other properties which relate addition and multiplication and call the resulting structure *association*.

The order relation in the ordered field F induces a natural order in the quotient class Q. We show that this is a total dense order relation compatible with the operations. If F is archimedean, the quotient class reduces to an ordered field. If F is non-archimedean, the quotient class contains magnitudes different from $\{0\}$ and its domain. Clearly $\{0\}$ is the minimal magnitude of Q, but Q has also a maximal magnitude which is its domain F itself; the minimal unity is $\{1\}$. In general, an association with these properties is called a *solid*. So we will prove that a quotient class of a non-archimedean field F is a solid. For the sake of clarity we give a full list of the axioms of a solid in the Appendix.

As remarked above, in solids distributivity does not hold in general. However, it turns out that in many cases full distributivity does hold, for example for elements of the same sign. Also it is possible to give necessary and sufficient conditions for the distributive law to hold for triples of elements of solids. The proofs are rather involved and are presented in a second paper.

In Section 2 we define the quotient class of an ordered field. We extend the order relation to the quotient class, prove that the property of trichotomy is maintained and show compatibility properties of the order with the algebraic operations. We recall also some basic notions of semigroups. In Section 3 we recall the notion of assembly which amounts to a regular semigroup with an idempotent condition on the magnitude operator. As a consequence the magnitude operator will be linear. We show that the quotient class is an assembly for addition and, leaving out the magnitudes, for multiplication. In Section 4 we define a structure called association which is, roughly speaking, a ring with individualized neutral elements for both addition and multiplication, and an adapted distributive law. Ordered associations are associations equipped with a total order relation respecting the algebraic operations. In Section 5 we define solids which are in a sense weakly distributive ordered fields with generalized neutral elements given by magnitudes and unities. We show that the quotient class of a non-archimedean field is a solid.

By the above, solids arise with non-archimedean fields. Archimedean solids may exist, but only in a set theory with a different axiomatics than conventional set theory. This question is briefly addressed at the end of the last section.

2. Quotient classes

Let $(F, +, \cdot, \leq, 0, 1)$ be a non-archimedean ordered field. Let $\mathcal{C}$ be the set of all convex subgroups for addition of F and Q be the set of all cosets with respect to the elements of $\mathcal{C}$. We will call the elements of $\mathcal{C}$ *magnitudes* and Q the *quotient class* of F with respect to $\mathcal{C}$. Observe that $\mathcal{C}$ is not reduced to $\{0\}$ and F itself. Indeed, a non-archimedean ordered field necessarily has infinitesimals other than 0. Let $\oslash$ denote the set of all infinitesimals in F. It is clearly convex and satisfies the group property, so $\oslash \in \mathcal{C}$. An element of $Q \setminus \mathcal{C}$ is called *zeroless*.

For $\alpha \in Q$, in the remainder of this section we use the notation $\alpha = a + A$, with $a \in F$ and $A \in \mathcal{C}$. Clearly A is unique but a is not. If α is zeroless, one proves by induction that $A/a \subseteq [-1/n, 1/n]$ for all $n \in \mathbb{N}$, hence $A/a \subseteq \oslash$.

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