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Fibonacci factoriangular numbers

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Abstract

Let $(F_m)_{m\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$, for all $m \geq 0$. In Castillo (2015), it is conjectured that 2, 5 and 34 are the only Fibonacci numbers of the form $n! + \frac{n(n+1)}{2}$, for some positive integer n. In this paper, we confirm the above conjecture.

Keywords: Fibonacci numbers; Factoriangular numbers; p-adic linear forms in logarithms of algebraic numbers

1. Introduction

The Fibonacci sequence $(F_m)_{m\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and

$$F_{m+2} = F_{m+1} + F_m \qquad \text{for all} \qquad m \geq 0.$$

The few terms of the Fibonacci sequence are

$$F := \{0, 1, 1, \mathbf{2}, 3, \mathbf{5}, 8, 13, 21, \mathbf{34}, 55, 89, 144, 233, 377, 610, \ldots\}.$$

Ljunggren [5] showed that the only squares in the Fibonacci sequence are 0, 1 and 144. This was rediscovered by Cohn [4] and Wyler [16]. London and Finkelstein [6] and Pethő [10] proved that the only cubes in the Fibonacci sequence are 0, 1 and 8. Bugeaud, Mignotte and Siksek [2]

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showed that the only perfect powers (of exponent larger than 1) in the Fibonacci sequence are 0, 1, 8 and 144. There are several other papers which study Diophantine equations arising from representing Fibonacci numbers by other quadratic and cubic polynomials such as $F_n = k^2+k+2$ (see [7]); $F_n = x^2 - 1$ or $F_n = x^3 \pm 1$ (see [11]); $F_n = px^2 + 1$ and $F_n = px^3 + 1$ for some fixed prime p (see [12]). Luca [8] proved that 55 is the largest number with only one distinct digit (called repdigit) in the Fibonacci sequence.

Recently, Castillo [3] dubbed a number of the form $Ft_n := n! + \frac{n(n+1)}{2}$ a *factoriangular* (from the sum between a factorial and the corresponding triangular). The first few factoriangular numbers are

$$Ft := \{\mathbf{2}, \mathbf{5}, 12, \mathbf{34}, 135, 741, 5068, 40356, 362925, \ldots\}.$$

This sequence is included in Sloane's The OnLine Encyclopedia of Integer Sequences (OEIS) [14] as sequence A101292. In [3], Castillo set forth the following conjecture.

Conjecture. *The only Fibonacci factoriangular numbers are* $F_3 = 2,\ F_5 = 5$ *and* $F_9 = 34$.

Here, we confirm Castillo's Conjecture.

Theorem 1. *The only Fibonacci factoriangular numbers are* 2, 5 *and* 34.

2. *p*-adic linear forms in logarithms

Our main tool is an upper bound for a non-zero p–adic linear form in two logarithms of algebraic numbers due to Bugeaud and Laurent [1].

We begin with some preliminaries. Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The *logarithmic height of* η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\max\{|\eta^{(i)}|, 1\}\right).$$

Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2 \in \mathbb{L} \setminus \{0, 1\}$ and b_1, b_2 positive integers. We put

$$\Lambda = \eta_1^{b_1} - \eta_2^{b_2}.$$

For a prime ideal π of the ring $\mathcal{O}_{\mathbb{L}}$ of algebraic integers in $\mathbb{L}$ and $\eta \in \mathbb{L}$, we denote by $\mathrm{ord}_\pi(\eta)$ the order at which π appears in the prime factorization of the principal fractional ideal $\eta\mathcal{O}_{\mathbb{L}}$ generated by η in $\mathbb{L}$. When η is an algebraic integer, $\eta\mathcal{O}_{\mathbb{L}}$ is an ideal of $\mathcal{O}_{\mathbb{L}}$. When $\mathbb{L} = \mathbb{Q}$, π is just a prime number. Let e_π and f_π be the ramification index and the inertial degree of π, respectively, and let $p \in \mathbb{Z}$ be the only prime number such that $\pi \mid p$. Then,

$$p\mathcal{O}_{\mathbb{L}} = \prod_{i=1}^{k} \pi_i^{e_{\pi_i}}, \quad |\mathcal{O}_{\mathbb{L}}/\pi| = p^{f_{\pi_i}} \quad \text{and} \quad d_{\mathbb{L}} = \sum_{i=1}^{k} e_{\pi_i} f_{\pi_i},$$

where $\pi_1 := \pi, \ldots, \pi_k$ are prime ideals in $\mathcal{O}_{\mathbb{L}}$.