INDAG: 471 Model 1 pp. 1–8 (col. fig: NIL)

ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Indagationes Mathematicae xx (xxxx) xxx–xxx

indagationes mathematicae

www.elsevier.com/locate/indag

A spectral characterization of Riesz homomorphisms between complex Riesz algebras

Fethi Benamor*, Ghaith Sellami

Research Laboratory of Algebra-Topology-Arithmetic-Order, Department of Mathematics, Faculty of Sciences of Tunis, Tunis-El Manar University, 2092 Tunis, Tunisia

Received 5 December 2016; received in revised form 24 March 2017; accepted 12 May 2017

Communicated by: B. de Pagter

Abstract

Let $\mathfrak{A}$ be a complex Riesz algebra with a positive identity e. We show that a (not necessary linear) functional $\phi : \mathfrak{A} \to \mathbb{C}$ is a unital Riesz homomorphism if and only if $\phi(\mathfrak{a}) - \phi(\mathfrak{b}) \in \sigma(P_e\mathfrak{a} - P_e\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$, where P_e denotes the order projection onto the center $\{e\}^{dd}$ of $\mathfrak{A}$. Then, as an application, we prove that unital Riesz homomorphisms, local unital Riesz homomorphisms, and 2-local Riesz homomorphisms between complex Riesz algebras with positive identities coincide.

Keywords: Complex Riesz algebra; Unital Riesz homomorphism; Spectrum; Local; 2-local

1. Introduction

Let $\mathfrak{A}$ be a complex algebra and $\phi : \mathfrak{A} \to \mathbb{C}$ be a linear functional. When $\mathfrak{A}$ is a unital Banach algebra, the Gleason–Kahane–Zelasko theorem asserts that ϕ is a nonzero algebra homomorphism if and only if $\phi(\mathfrak{a}) \in \sigma(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$. See [3,6]. Assume now that $\mathfrak{A}$ is a unital Banach f-algebra. It follows from [5] that ϕ is a nonzero algebra homomorphisms if and only if ϕ is a unital Riesz homomorphism. So we can assert that ϕ is a unital Riesz homomorphism if and only if $\phi(\mathfrak{a}) \in \sigma(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$. The case when $\mathfrak{A}$ is just a Banach Riesz algebra with a positive identity was investigated by Huijssman [4]. He shows that in such a case ϕ is

* Corresponding author.
E-mail addresses: fethi.benamor@ipest.rnu.tn (F. Benamor), ghaith99@hotmail.fr (G. Sellami).

http://dx.doi.org/10.1016/j.indag.2017.05.004

Please cite this article in press as: F. Benamor, G. Sellami, A spectral characterization of Riesz homomorphisms between complex Riesz algebras, Indagationes Mathematicae (2017), http://dx.doi.org/10.1016/j.indag.2017.05.004.

a unital Riesz homomorphism if and only if $\phi(\mathfrak{a}) \in \sigma(P_e\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$, where P_e denotes the order projection (also called band projection) on $\{e\}^{dd}$. Recently, Azouzi and Boulabiar [1] studied Gleason–Kahane–Zelasko type theorems for unital Riesz algebras. They show that the result by Huijssman remains true when $\mathfrak{A}$ is a complex (not necessary Banach) Riesz algebra with a positive identity.

In [7], Kowalsky and Slodowsky extended the Gleason–Kahane–Zelasko theorem to nonlinear functionals. More precisely, they proved that if $\mathfrak{A}$ is a unital complex Banach algebra and $\phi : \mathfrak{A} \to \mathbb{C}$ is a (not necessary linear) functional, then ϕ is a nonzero algebra homomorphism if and only if $\phi(0) = 0$ and $\phi(\mathfrak{a}) - \phi(\mathfrak{b}) \in \sigma(\mathfrak{a} - \mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. The main purpose of this work is to give a Riesz algebra version of the Kowalsky–Slodowsky theorem. Our main result will be applied to study local and 2-local Riesz homomorphisms on complex Riesz algebras.

2. Preliminaries

Let $\mathfrak{A}$ be a *complex Riesz space*, that is, $\mathfrak{A}$ is the complexification of a uniformly complete (real) Riesz space $\mathfrak{A}_\mathbb{R}$. Observe that each $\mathfrak{a} \in \mathfrak{A}$ can be uniquely written in the form $\mathfrak{a} = \operatorname{Re}\mathfrak{a} + i \operatorname{Im}\mathfrak{a}$ where $\operatorname{Re}\mathfrak{a}, \operatorname{Im}\mathfrak{a} \in \mathfrak{A}_\mathbb{R}$. Recall also that $\mathfrak{A}$ is endowed with the standard *absolute value* defined by

$$|\mathfrak{a}| = \sup\{(\cos\theta)\operatorname{Re}\mathfrak{a} + (\sin\theta)\operatorname{Im}\mathfrak{a} : \theta \in [0, 2\pi]\}, \text{ for all } \mathfrak{a} \in \mathfrak{A}.$$

Evidently, $|\operatorname{Re}\mathfrak{a}| \le |\mathfrak{a}|$ and $|\operatorname{Im}\mathfrak{a}| \le |\mathfrak{a}|$ for all $\mathfrak{a} \in \mathfrak{A}$. An element $\mathfrak{a} \in \mathfrak{A}$ is said to be *real* whenever $\operatorname{Im}\mathfrak{a} = 0$, and *positive* whenever $\operatorname{Im}\mathfrak{a} = 0$ and $\operatorname{Re}\mathfrak{a} \ge 0$.

Now, let $\mathfrak{A}$ and $\mathfrak{B}$ be two complex Riesz spaces, and $T : \mathfrak{A} \to \mathfrak{B}$ be a linear operator. We say that T is *real* (respectively, *positive*) whenever T maps real (respectively, positive) elements in $\mathfrak{A}$ into real (respectively, positive) elements in $\mathfrak{B}$. Also we call T a *Riesz homomorphism* if $|T\mathfrak{a}| = T|\mathfrak{a}|$ for all $\mathfrak{a} \in \mathfrak{A}$. We can prove that T is a Riesz homomorphism if and only if T is real and $x \wedge y = 0$ implies $T(x) \wedge T(y) = 0$ for all real elements $x, y \in \mathfrak{A}$.

A complex Riesz space $\mathfrak{A}$ will be said a *complex Riesz algebra* if there exists an associative multiplication in $\mathfrak{A}$ with the usual algebra properties such that $|\mathfrak{a}\mathfrak{b}| \le |\mathfrak{a}|\,|\mathfrak{b}|$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. A complex Riesz algebra $\mathfrak{A}$ for which

$$|\mathfrak{a}| \wedge |\mathfrak{b}| = 0 \text{ implies } (|\mathfrak{c}|\,|\mathfrak{a}|) \wedge |\mathfrak{b}| = (|\mathfrak{a}|\,|\mathfrak{c}|) \wedge |\mathfrak{b}| = 0 \text{ for all } \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{A} \quad (1)$$

is called a *complex f-algebra*. The complex Riesz space $\mathfrak{A}$ is a complex Riesz algebra if and only if $\mathfrak{A}_\mathbb{R}$ is a Riesz algebra. Also $\mathfrak{A}$ is a complex f-algebra if and only if $\mathfrak{A}_\mathbb{R}$ is an f-algebra.

Assume now that $\mathfrak{A}$ is a complex Riesz algebra with a positive identity e. A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be a *f-subalgebra* of $\mathfrak{A}$ whenever $\mathfrak{B}$ is a Riesz subalgebra of $\mathfrak{A}$ that satisfies the condition (1). Moreover $\mathfrak{B}$ is said to be *full* whenever $\mathfrak{B}$ is closed under inversion, that is, for any $\mathfrak{a} \in \mathfrak{B}$ which has an inverse a^{-1} in $\mathfrak{A}$ we have $a^{-1} \in \mathfrak{B}$. It turns out that the principal band $\{e\}^{dd}$ generated by e in $\mathfrak{A}$ is a full f-subalgebra of $\mathfrak{A}$. Thus, for any $\mathfrak{a} \in \{e\}^{dd}$, the spectrum $\sigma(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{A}$ coincides with its spectrum $\sigma_e(\mathfrak{a})$ in $\{e\}^{dd}$. The principal band $\{e\}^{dd}$ is also a projection band in $\mathfrak{A}$, that is $\mathfrak{A} = \{e\}^{dd} \oplus \{e\}^{d}$. So, if P_e denotes the order projection onto $\{e\}^{dd}$, then $\sigma(P_e\mathfrak{a}) = \sigma_e(P_e\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$.

At this point we consider two complex Riesz algebras $\mathfrak{A}$ and $\mathfrak{B}$ with positive identities $e_\mathfrak{A}$ and $e_\mathfrak{B}$ respectively and a linear operator $T : \mathfrak{A} \to \mathfrak{B}$. We say that T is *unital* whenever $T(e_\mathfrak{A}) = e_\mathfrak{B}$. Also T is called an *algebra homomorphism* if $T(\mathfrak{a}\mathfrak{b}) = T(\mathfrak{a})T(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. When $\mathfrak{A}$ and $\mathfrak{B}$ are complex f-algebras, unital Riesz homomorphisms coincide with unital algebra homomorphisms.

Please cite this article in press as: F. Benamor, G. Sellami, A spectral characterization of Riesz homomorphisms between complex Riesz algebras, Indagationes Mathematicae (2017), http://dx.doi.org/10.1016/j.indag.2017.05.004.