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Enlargement of subgraphs of infinite graphs by Bernoulli percolation

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Abstract

We consider changes in properties of a subgraph of an infinite graph resulting from the addition of open edges of Bernoulli percolation on the infinite graph to the subgraph. We give the triplet of an infinite graph, one of its subgraphs, and a property of the subgraphs. Then, in a manner similar to the way Hammersley's critical probability is defined, we can define two values associated with the triplet. We regard the two values as certain critical probabilities, and compare them with Hammersley's critical probability. In this paper, we focus on the following cases of a graph property: being a transient subgraph, having finitely many cut points or no cut points, being a recurrent subset, or being connected. Our results depend heavily on the choice of the triplet.

Most results of this paper are announced in Okamura (2016) [24] without proofs. This paper gives full details of them.

Keywords: Bernoulli percolation; Critical probability; Transient graphs

1. Introduction and main results

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A connected graph is called transient (resp. recurrent) if the simple random walk on it is transient (resp. recurrent). Benjamini, Gurel-Gurevich and Lyons [4] showed the cerebrating

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result claiming that the trace of the simple random walk on a transient graph is recurrent almost surely. If a connected subgraph of an infinite connected graph is transient, then the infinite connected graph is transient. Therefore, the trace is somewhat "smaller" than the graph on which the simple random walk runs. Now we consider the following questions: How far are a transient graph G and the trace of the simple random walk on G? More generally, how far are G and a recurrent subgraph H of G? How many edges of G do we need to add to H so that the enlargement of H becomes transient?

There are numerous choices of edges of G to be added to H. If we add finitely many edges to H, then the enlarged graph is also recurrent. Therefore, we add *infinitely* many edges to H and consider whether the enlarged graph is transient. In this paper, we add infinitely many edges of G to H *randomly*. Specifically, we add open edges of Bernoulli bond percolation on G to H, and consider the probability that the enlargement of H is transient.

Now we state our framework. In this paper, a graph is a locally-finite simple graph. A simple graph is a non-oriented graph in which neither multiple edges or self-loops are allowed. $V(X)$ and $E(X)$ denote the sets of vertices and edges of a graph X, respectively. Denote the cardinality of $A \subset V(X)$ by $|A|$. If we consider the d-dimensional integer lattice $\mathbb{Z}^d$, then it is the nearest-neighbor model.

Let G be an infinite connected graph. We say that a subgraph H of G is *connected* if for any two vertices x and y of H there are vertices $x_0, \dots, x_n$ of H such that $x_0 = x$, $x_n = y$, and $\{x_{i-1}, x_i\} \in E(H)$ for each i. In this paper, we consider Bernoulli *bond* percolation and do not consider site percolation. Let $\mathbb{P}_p$ be the Bernoulli measure on the space of configurations of Bernoulli bond percolation on G such that each edge of G is open with probability $p \in (0, 1)$. Denote a configuration of percolation by $\omega = (\omega_e)_{e \in E(G)} \in \{0, 1\}^{E(G)}$. We say that an edge e is open if $\omega_e = 1$ and closed otherwise. We say that an event $A \subset \{0, 1\}^{E(G)}$ is increasing (resp. decreasing) if the following holds: if $\omega = (\omega_e) \in A$ and $\omega'_e \geq \omega_e$ (resp. $\omega'_e \leq \omega_e$) for any $e \in E(G)$, then $\omega' \in A$. Let C_x be the open cluster containing $x \in V(G)$. We remark that $\{x\} \subset V(C_x)$ holds. By convention, we often denote the set of vertices $V(C_x)$ by C_x. Consider the probability that the number of vertices of G connected by open edges from a fixed vertex is infinite under $\mathbb{P}_p$. Then Hammersley's critical probability $p_c(G)$ is the infimum of p such that the probability is positive, that is, for some $x \in V(G)$,

$$p_c(G) = \inf\left\{p \in (0, 1) : \mathbb{P}_p(|C_x| = +\infty) > 0\right\}.$$

This value does not depend on the choice of x.

Similarly, we consider the probability that the enlarged graph is transient under $\mathbb{P}_p$ and either of the following two values: the infimum of p such that the probability is positive, or the infimum of p such that the probability is one. We regard these two values as certain critical probabilities, and compare them with Hammersley's critical probability. We also consider questions of this kind, not only for transience, but also for other graph properties.

Definition 1.1 (*Enlargement of Subgraph*). Let H be a subgraph of G. Let $\mathcal{U}(H) = \mathcal{U}_\omega(H)$ be a random subgraph of G such that

$$V(\mathcal{U}(H)) := \bigcup_{x \in V(H)} V(C_x) \text{ and } E(\mathcal{U}(H)) := E(H) \cup \left(\bigcup_{x \in V(H)} E(C_x) \right).$$

If ω is chosen according to $\mathbb{P}_p$, then, we write $\mathcal{U}(H) = \mathcal{U}_p(H) = \mathcal{U}_{p,\omega}(H)$.