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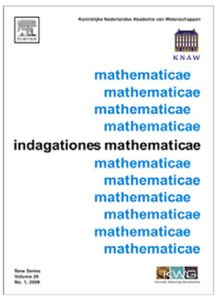

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HARNACK AND SHMUL'YAN PRE-ORDER RELATIONS FOR HILBERT SPACE CONTRACTIONS

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Abstract. We study the behavior of some classes of Hilbert space contractions with respect to Harnack and Shmul'yan pre-orders and the corresponding equivalence relations. We give some conditions under which the Harnack equivalence of two given contractions is equivalent to their Shmul'yan equivalence and to the existence of an arc joining the two contractions in the class of operator-valued contractive analytic functions on the unit disc. We apply some of these results to quasi-isometries and quasi-normal contractions, as well as to partial isometries for which we show that their Harnack and Shmul'yan parts coincide. We also discuss an extension, recently considered by S. ter Horst [*J. Operator Th. 72(2014), 487–520*], of the Shmul'yan pre-order from contractions to the operator-valued Schur class of functions. In particular, the Shmul'yan-ter Horst part of a given partial isometry, viewed as a constant Schur class function, is explicitly determined.

1. Introduction

Preamble. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}_1(\mathcal{H})$ denote the unit ball of the C*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Following the usual terminology of operator theory, elements of $\mathcal{B}_1(\mathcal{H})$ are called contractions. One tool in the study of the (hyperbolic) geometry of $\mathcal{B}_1(\mathcal{H})$ is the use of order relations such as the Harnack and Shmul'yan pre-orders. Both pre-order relations have nice geometric and analytic interpretations. Although these two pre-orders have been around since 1970s and 1980s [1,5,15–17,20,24], their structure is to date not completely understood, and in recent years there has been an increase in interest for this topic [2,7,8,10,12,13,18,19,23].

The aim of this paper is to study the behavior of some classes of Hilbert space contractions with respect to the Harnack and Shmul'yan pre-orders, and the corresponding equivalence relations. We look at contractions that also have certain commutativity properties, for instance we consider the cases where two operators are commuting, doubly commuting or when the operators are themselves quasi-normal or hyponormal. The case of partial isometries is thoroughly analyzed and the role of commutativity properties is discussed.

Notation and basic definitions. In this paper $T, T' \in \mathcal{B}(\mathcal{H})$ will be linear contractions acting on the complex Hilbert space $\mathcal{H}$. Also, V acting on $\mathcal{K} \supset \mathcal{H}$ and V' acting on $\mathcal{K}' \supset \mathcal{H}'$ will denote the *minimal isometric dilations* of T and T' respectively. Recall (see [25]) that V is a minimal isometric dilation of T if V is an isometry on $\mathcal{K} = \bigvee_{n\geq 0} V^n\mathcal{H}$ satisfying

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