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A globalization of a theorem of Horozov

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Abstract

This paper proves a globalization of a theorem of Horozov (1990). In particular, it determines the inverse of the Horozov map from an explicitly determined simply connected domain in the upper half plane to the set of regular values of the energy–momentum map of the spherical pendulum with the h-axis removed for $h \geq 1$.

Keywords: Spherical pendulum; Horozov's theorem; Global diffeomorphism

1. Introduction

In the study of action angle coordinates for the spherical pendulum [1, Chpt. IV] we come across the real analytic functions

$$2\pi\ \Theta(h,\ell) = 2\ell \int_{x_-}^{x_+} \frac{1}{(1-x^2)\sqrt{2(h-x)(1-x^2)-\ell^2}}\, \mathrm{d}x \tag{1}$$

and

$$T(h,\ell) = 2 \int_{x_-}^{x_+} \frac{1}{\sqrt{2(h-x)(1-x^2)-\ell^2}}\, \mathrm{d}x, \tag{2}$$

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where $-1 < x_- < x_+ < 1$ are real roots of $2(h-x)(1-x^2)-\ell^2$, when (h,ℓ) lies in

$$\mathrm{R} = \Big\{ (h,\ell) \in \mathbb{R}^2 \; -1 < h,\ \frac{27}{4}\ell^2 < (3+h^2)^{3/2} + h(9-h^2)\ \&\ (h,\ell) \neq (1,0) \Big\}.$$

R is the set of regular values of the energy–momentum mapping of the spherical pendulum. Geometrically, let $\mathcal{C}$ be a cross section to the motion of the spherical pendulum of energy h and angular momentum ℓ. Then $T(h,\ell)$ is the time of first return to $\mathcal{C}$ of a motion of the spherical pendulum starting on $\mathcal{C}$; whereas $\Theta(h,\ell)$ is the rotation number of this motion.

The theorem of Horozov [4], see also [3], states that for every $(h,\ell) \in \mathrm{R}$ we have $\det \begin{pmatrix} \frac{\partial T}{\partial h} & \frac{\partial \Theta}{\partial h} \\ \frac{\partial T}{\partial \ell} & \frac{\partial \Theta}{\partial \ell} \end{pmatrix} < 0$. In other words, Horozov's mapping

$$\Psi : \mathrm{R} \to \mathbb{R}^2 : (h,\ell) \mapsto \big(\Theta(h,\ell), T(h,\ell)\big) \tag{3}$$

is a *local* real analytic diffeomorphism. The goal of this paper is to prove the following globalization of Horozov's theorem.

Theorem. *The global Horozov mapping*

$$\begin{aligned} \widetilde{\Psi} : \mathrm{R} \setminus \{h > 1 \& \ell = 0\} \to \mathcal{D} \subseteq (0,1)\times(0,\infty) : (h,\ell) \\ \mapsto \begin{cases} \big(\Theta(h,\ell), T(h,\ell)\big), & \text{if } (h,\ell) \in (\mathrm{R} \cap \{\ell \geq 0\}) \setminus \{h > 1\ \&\ \ell = 0\} \\ \big(\Theta(h,\ell)+1, T(h,\ell)\big), & \text{if } (h,\ell) \in (\mathrm{R} \cap \{\ell \leq 0\}) \setminus \{h > 1\ \&\ \ell = 0\} \end{cases} \end{aligned} \tag{4}$$

is a real analytic diffeomorphism. A precise description of the simply connected domain $\mathcal{D}$ is given in Corollary 4.3.

Since $\mathrm{R} \setminus \{h > 1 \& \ell = 0\}$ is simply connected, the map Ψ is just one sheet of a real analytic universal covering map of R.

2. Local properties of Θ and T

In this section we show that on R the functions Θ and T are locally real analytic.

First we show

Lemma 2.1. *The functions Θ (1) and T (2) are locally real analytic on $\mathrm{R}^\vee = \mathrm{R} \setminus \{\ell = 0\}$.*

Proof. First we treat Θ. Let $P(z) = 2(h-z)(1-z^2)-\ell^2$, where $(h,\ell) \in \mathrm{R}^\vee$. Consider the 1-form $\varpi = \frac{1}{(1-z^2)\sqrt{P(z)}}\, dz$ on $\mathbb{C}^\vee$, the extended complex plane, which is cut along the real axis between x_- and x_+ and again between x_0 and ∞. Here $x_{\pm,0}$ are distinct roots of P with

$$\begin{cases} -1 < x_- < x_+ < h < 1 < x_0, & \text{if } -1 < h < 1 \\ -1 < x_- < x_+ < 1 < h < x_0, & \text{if } h > 1. \end{cases}$$

Write $\sqrt{P(z)} = \sqrt{r_- r_+ r_0}\, e^{i(\theta_- + \theta_+ + \theta_0)/2}$, where $z - x_{0,\pm} = r_{0,\pm} e^{i\theta_{0,\pm}}$ and $0 \leq \theta_{0,\pm} < 2\pi$. With this choice of complex square root on $\mathbb{C}^\vee$ we see that ϖ is single-valued and meromorphic with a first order pole at $z = \pm 1$ whose residue is

$$\begin{aligned} \operatorname*{Res}_{z=\pm 1} \varpi &= \lim_{z \to \pm 1} (z \mp 1) \frac{1}{(1-z^2)\sqrt{P(z)}} \\ &= -\lim_{z\to\pm 1} \frac{1}{z \pm 1} \frac{1}{\sqrt{P(z)}} = \mp \frac{1}{2\sqrt{P(\pm 1)}} = i\, \frac{1}{2|\ell|}. \end{aligned} \tag{5}$$