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A Mandelpinski maze for rational maps of the form $z^n + \lambda/z^d$

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Dedicated to Henk Broer on the occasion of his 65th Birthday

Abstract

In this paper we identify a new type of structure that lies in the parameter plane of the family of maps $z^n + \lambda/z^d$ where $n \geq 2$ is even but $d \geq 3$ is odd. We call this structure a Mandelbrot–Sierpinski maze. Basically, the maze consists at the first level of an infinite string of alternating Mandelbrot sets and Sierpinski holes that lie along an arc in the parameter plane for this family. At the next level, there are infinitely many smaller Mandelbrot sets and Sierpinski holes that alternate on the arc between each Mandelbrot set and Sierpinski hole on the previous level, and then finitely many other Mandelbrot sets and Sierpinski holes that extend away from the given Mandelbrot set in a pair of different directions. And then this structure repeats inductively to produce the "Mandelpinski" maze.

Keywords: Julia sets; Mandelpinski maze; Rational maps

In this paper we will concentrate on the family of maps $F_\lambda(z) = z^n + \lambda/z^d$ where $n, d \geq 2$. It is known that there are several different and very interesting geometric structures surrounding the negative real axis in the parameter planes for these maps. For example, when n and d are even, it has been shown in [1] that there is a "Cantor necklace" that lies along the negative real axis in the parameter plane and a "principal" Mandelbrot set along the positive axis. A Cantor necklace is a set that is a continuous image of the Cantor middle-thirds set to which is adjoined countably many open disks in the plane in place of the removed open intervals along the real

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line. For parameters inside these open disks (which we call Sierpinski holes), the Julia set of F_λ is known to be a Sierpinski curve (i.e., is homeomorphic to the Sierpinski carpet fractal), and the different dynamical behaviors on these Julia sets is completely understood [12]. When n is odd and d is even, there is no such Cantor necklace; rather there are now two "principal" Mandelbrot sets, one along the positive real axis and the other along the negative real axis. As a consequence, the dynamical behavior for these parameters is very different from the behavior when n and d is even. Thus the remaining case is when n is even and d is odd; this too is a very different case that we shall deal with in this paper.

As when d is even, we again have a principal Mandelbrot set straddling the positive real axis. But the structure on and around the negative real axis is very different. We shall show that there is a "Mandelpinski maze" (an MS-maze) in a neighborhood of the negative real axis in the parameter plane. Roughly speaking, this is a set that consists of infinitely many baby Mandelbrot sets and Sierpinski holes that alternate along a specific planar graph that has infinitely many vertices.

1. Preliminaries

In this paper we consider the family of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n \geq 2$ is even and $d \geq 3$ is odd. When $|z|$ is large, we have that $|F_\lambda(z)| > |z|$, so the point at ∞ is an attracting fixed point in the Riemann sphere. We denote the immediate basin of attraction of ∞ by B_λ. There is also a pole at the origin for each of these maps, and so there is a neighborhood of the origin that is mapped into B_λ. If the preimage of B_λ surrounding the origin is disjoint from B_λ, we call this region the trap door and denote it by T_λ.

The Julia set of F_λ, $J(F_\lambda)$, has several equivalent definitions. $J(F_\lambda)$ is the set of all points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ, and it is also the boundary of the set of all points whose orbits tend to ∞ under iteration of F_λ, not just those in the boundary of B_λ. See [11].

One checks easily that there are $n + d$ critical points that are given by

$$c^\lambda = \left(\frac{d\lambda}{n}\right)^{\frac{1}{n+d}}$$

with the corresponding critical values given by

$$v^\lambda = \frac{(d+n)\lambda^{\frac{n}{n+d}}}{d^{\frac{d}{n+d}} n^{\frac{n}{n+d}}}.$$

There are also $n + d$ prepoles given by

$$p^\lambda = (-\lambda)^{\frac{1}{n+d}}.$$

The straight ray extending from the origin to ∞ and passing through the critical point c^λ is called the *critical point ray*. This ray is mapped two-to-one onto the portion of the straight ray from the origin to ∞ that starts at the critical value $F_\lambda(c^\lambda)$ and extends to ∞ beyond this critical value. A similar straight line extending from 0 to ∞ and passing through a prepole p^λ is a *prepole ray*,

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