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Exotic topology in complex dynamics

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Dedicated to Henk Broer on the occasion of his 65th Birthday

Abstract

We describe three different exotic topological objects that arise as Julia sets for complex maps, namely, Cantor bouquets, indecomposable continua, and Sierpinski curves.

Keywords: Julia set; Complex dynamics; Cantor bouquet; Indecomposable continuum; Sierpinski curve

1. Introduction

In planar topology, there are many objects that are quite "strange", at least to people who are not topologists. These sets are very interesting, from a topological point of view, and quite beautiful. But these sets seem to be not the kind of thing you would encounter in a typical topological situation; rather, they seem to be very special counterexamples to theorems, not the objects that you run into in everyday life. Interestingly, since the rebirth of the field of complex dynamics in the 1980s, many of these objects have now reappeared as the Julia sets for complex analytic functions. Moreover, they appear all the time in this setting.

In this paper, we shall give three examples of these exotic topological spaces, namely, Cantor bouquets, indecomposable continua, and Sierpinski curves, and we shall show how they arise in specific families of complex maps, including the complex exponential family and a particular family of singularly perturbed rational maps.

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2. Julia sets

Let $F : \mathbb{C} \to \mathbb{C}$ be a complex analytic function, and let F^n denote F composed with itself n times, the nth iterate of F. For a point $z \in \mathbb{C}$, the *orbit* of z is the sequence $z, F(z), F^2(z), \dots$. Of interest in dynamics is the fate of these orbits: is this fate predictable or is it not?

In complex dynamics, the predictable set is the *Fatou set*; points in this set have the property that all nearby orbits behave "similarly". Thanks to work of Julia and Fatou in the 1910s and Sullivan in the 1980s, the dynamics of F on the Fatou set is completely understood. There are only a few types of behaviors associated with such points: most often, points in the Fatou set simply tend to an attracting periodic orbit. This is the orbit of a point z for which $F^n(z) = z$ and $|(F^n)'(z)| < 1$. So there is an open neighborhood about z in which all points have orbits that tend to the periodic orbit. There are a few other types of Fatou components which we shall not deal with in this paper, but attracting basins are by far the most common types of Fatou components.

The *Julia set* of F, denoted by $J(F)$, is the complement of the Fatou set: it consists of points for which nearby orbits behave in vastly different manners. This is the "chaotic" set for such maps. By a classical theorem of Montel, if z is a point in the Julia set of F and U is any neighborhood of z, then the union of the forward images of U contains the entire plane (with the exception of at most one point). So F depends quite sensitively on initial conditions on its Julia set in the sense that a small error in specifying the initial point can lead to huge changes in the fate of the orbit. There are other equivalent definitions of the Julia set. For example, the Julia set is also the closure of the set of repelling periodic points for F, i.e., periodic points z for which $F^n(z) = z$ and $|(F^n)'(z)| > 1$. From a dynamical systems point of view, all of the interesting behavior of a complex analytic function occurs on its Julia set, and it is this set that often exhibits the interesting topology.

As a simple example, consider the function $F(z) = z^2$. The behavior of all orbits of this function is easy to describe. If $|z| < 1$, then $|F(z)| < |z|$ and so all orbits that begin inside the unit circle simply tend to 0, which is an attracting fixed point. If $|z| > 1$, then all orbits increase in magnitude and tend to ∞. Finally, if z lies on the unit circle, then the images of any small neighborhood of this point under F^n eventually cover the entire plane, except (possibly) the origin. As a consequence, the Julia set of z^2 is the unit circle, and the Fatou set contains all other points in $\mathbb{C}$. The reader should be forewarned that very few other Julia sets are as simple to understand. Most often, these Julia sets are extremely complicated fractal sets with very complicated topology.

As we shall see below, there are other definitions of the Julia set depending upon the type of complex map involved. For example, if F is a complex polynomial, then the point at ∞ in the Riemann sphere is always an attracting fixed point, so we have a basin of attraction of this fixed point. Then $J(F)$ is the boundary of this basin of attraction.

Such an attracting fixed point at ∞ does not necessarily occur for rational maps or entire functions, so this definition does not apply in these cases. However, for the complex exponential function $\lambda \exp(z)$, it is known that the Julia set is now the closure of (not the boundary of) the set of points that escape to ∞ [17].

3. Cantor bouquets and the complex exponential

Our first example of an interesting (and crazy) Julia set is a *Cantor bouquet*. Roughly speaking, a Cantor bouquet is an uncountable collection of disjoint continuous curves tending to ∞ in a certain direction in the plane, each of which has a distinguished endpoint. More precisely,

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