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The Poincaré method: A powerful tool for analyzing synchronization of coupled oscillators

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Highlights

- A mathematical framework for studying synchronization is presented.
- The framework is based on the Poincaré method of small parameter.
- The proposed methodology is applicable to weakly nonlinear coupled systems.
- For the sake of illustration a particular example is analyzed.

Abstract

This paper focuses on the application of the Poincaré method of 'small parameter' for the study of coupled dynamical systems. Specifically, our attempt here is to show that, by using the Poincaré method, it is possible to derive conditions for the onset of synchronization in coupled (oscillatory) systems. A case of study is presented, in which conditions for the existence and stability of synchronous solutions, occurring in two nonlinear oscillators interacting via delayed dynamic coupling, are derived. Ultimately, it is demonstrated that the Poincaré method is indeed an effective tool for analyzing the synchronous behavior observed in coupled dynamical systems.

Keywords: Poincaré method; Coupled systems; Synchronization; Time-delay

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http://dx.doi.org/10.1016/j.indag.2015.11.008
0019-3577/© 2015 Published by Elsevier B.V. on behalf of Royal Dutch Mathematical Society (KWG).

1. Introduction

Around 1890, the 'impatient genius' and French scientist Jules Henri Poincaré (1854–1912), unveiled a mathematical setting for analyzing dynamical systems [19,24,3]. In particular, Poincaré developed an analytic method for determining the approximated solutions of a nonlinear system. Although the mathematical framework of Poincaré was originally used in the study of celestial mechanics, it should be emphasized that his methodology can be used in other areas, like in the study of periodic solutions and bifurcations occurring in certain dynamical systems, see, e.g. [25,7,6,9,8].

The mathematical machinery constructed by Poincaré is of utmost importance if one considers the fact that, in general, explicit solutions of nonlinear dynamical systems cannot be obtained. In fact, many physical, chemical, biological, and engineering problems are described by nonlinear differential equations, which analysis is far from trivial. In many cases, however, an approximated solution – which approximately describes the real behavior of the system – may be obtained by using the Poincaré method [1].

The key requirement of the Poincaré method is that there exists a 'small' parameter, say μ, such that for $\mu = 0$, a known solution exists. Once the solution of the 'unperturbed' system is known, the Poincaré method searches for a solution of the original system, i.e. $\mu \neq 0$, in the form of a power series. This was formally established in the Poincaré expansion theorem, which guarantees that in a neighborhood of the 'unperturbed' solution, a convergent expansion can be obtained with respect to the small parameter. Poincaré proved this by using majorization of series in his Méthodes Nouvelles [20]. A modern and shorter proof of Poincaré's theorem, using advanced complex function theory, has been presented in [25].

Nowadays, there exist several perturbation techniques for analyzing nonlinear systems, see, e.g. [26,18,4,5,10,11,21]. All of them aiming to obtain the approximated solution of a nonlinear dynamical system, taking as a reference a 'well-known' solution.

In this note, a mathematical framework, which is based on the classical Poincaré method of 'small parameter', is reviewed. Originally, the framework was introduced by Malkin and Blekhman in [16,2] and was applied in the study of periodic solutions occurring in weakly nonlinear systems. Here, it is shown that the framework can be used to determine the onset of synchronization in coupled dynamical systems.

In particular, it is demonstrated that the aforementioned methodology allows to derive necessary and sufficient conditions for the existence and stability of synchronous solutions in *weakly* nonlinear systems with weak coupling.

An application example, namely a pair of self-sustained nonlinear oscillators, interacting via delayed dynamic coupling, is presented. Two types of synchronization are investigated as a function of the time-delay, namely in-phase and anti-phase synchronization. Additionally, the amplitude and frequency of the synchronous solutions are analytically obtained.

The rest of the manuscript is organized as follows. First, in Section 2, the Poincaré method for weakly nonlinear systems is reviewed. Then, an application example is presented in Section 3. Namely, conditions for synchronization of a pair of nonlinear oscillators, interacting via delayed dynamic coupling, are derived. Finally, a discussion of the obtained results is provided in Section 4.

2. The Poincaré method for weakly nonlinear systems

Consider the weakly nonlinear system

$$\dot{x} = Ax + \mu \Phi(x), \tag{1}$$

Please cite this article in press as: J. Pena Ramirez, H. Nijmeijer, The Poincaré method: A powerful tool for analyzing synchronization of coupled oscillators, Indagationes Mathematicae (2015), http://dx.doi.org/10.1016/j.indag.2015.11.008